Australian Intermediate Mathematics Olympiad 2017

Questions

1. The number x is 111 when written in base b, but it is 212 when written in base $b-2$. What is x in base 10?

[2 marks]

2. A triangle ABC is divided into four regions by three lines parallel to BC. The lines divide AB into four equal segments. If the second largest region has area 225, what is the area of ABC?

[2 marks]

3. Twelve students in a class are each given a square card. The side length of each card is a whole number of centimetres from 1 to 12 and no two cards are the same size. Each student cuts his/her card into unit squares (of side length 1 cm). The teacher challenges them to join all their unit squares edge to edge to form a single larger square without gaps. They find that this is impossible.

Alice, one of the students, originally had a card of side length a cm. She says, 'If I don't use any of my squares, but everyone else uses their squares, then it is possible!'

Bob, another student, originally had a card of side length b cm. He says, 'Me too! If I don't use any of my squares, but everyone else uses theirs, then it is possible!'

Assuming Alice and Bob are correct, what is ab?

[3 marks]

4. Aimosia is a country which has three kinds of coins, each worth a different whole number of dollars. Jack, Jill, and Jimmy each have at least one of each type of coin. Jack has 4 coins totalling \$28, Jill has 5 coins worth \$21, and Jimmy has exactly 3 coins. What is the total value of Jimmy's coins?

[3 marks]

5. Triangle ABC has $AB = 90$, $BC = 50$, and $CA = 70$. A circle is drawn with centre P on AB such that CA and CB are tangents to the circle. Find $2AP$.

[3 marks]

6. In quadrilateral $PQRS$, $PS = 5$, $SR = 6$, $RQ = 4$, and $\angle P = \angle Q = 60°$. Given that $2PQ = a + \sqrt{b}$, where a and b are unique positive integers, find the value of $a + b$.

[4 marks]

PLEASE TURN OVER THE PAGE FOR QUESTIONS 7, 8, 9 AND 10

7. Dan has a jar containing a number of red and green sweets. If he selects a sweet at random, notes its colour, puts it back and then selects a second sweet, the probability that both are red is 105% of the probability that both are red if he eats the first sweet before selecting the second. What is the largest number of sweets that could be in the jar?

[**4 marks**]

8. Three circles, each of diameter 1, are drawn each tangential to the others. A square enclosing the three circles is drawn so that two adjacent sides of the square are tangents to one of the circles and the square is as small as possible. The side length of this square is $a + \dfrac{\sqrt{b} + \sqrt{c}}{12}$ where a, b, c are integers that are unique (except for swapping b and c). Find $a + b + c$.

[**4 marks**]

9. Ten points $P_1, P_2, \ldots, P_{10}$ are equally spaced around a circle. They are connected in separate pairs by 5 line segments. How many ways can such line segments be drawn so that only one pair of line segments intersect?

[**5 marks**]

10. *Ten-dig* is a game for two players. They try to make a 10-digit number with all its digits different. The first player, A, writes any non-zero digit. On the right of this digit, the second player, B, then writes a digit so that the 2-digit number formed is divisible by 2. They take turns to add a digit, always on the right, but when the nth digit is added, the number formed must be divisible by n. The game finishes when a 10-digit number is successfully made (in which case it is a *draw*) or the next player cannot legally place a digit (in which case the other player *wins*).

Show that there is only one way to reach a draw.

[**5 marks**]

Investigation

Show that if A starts with any non-zero even digit, then A can always win no matter how B responds.

[**4 bonus marks**]

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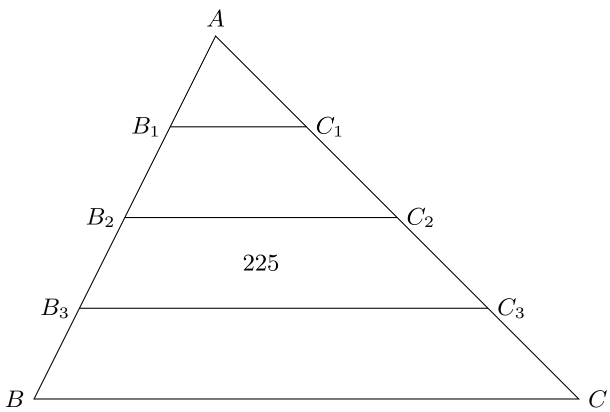
Solutions

1. We have $x = b^2 + b + 1$ and $x = 2(b-2)^2 + (b-2) + 2 = 2(b^2 - 4b + 4) + b = 2b^2 - 7b + 8$. $\boxed{1}$
Hence $0 = (2b^2 - 7b + 8) - (b^2 + b + 1) = b^2 - 8b + 7 = (b-7)(b-1)$.
From the given information, $b - 2 > 2$. So $b = 7$ and $x = 49 + 7 + 1 = \mathbf{57}$. $\boxed{1}$

2. *Method 1*

Let B_1C_1, B_2C_2, B_3C_3, be the lines parallel to BC as shown. Then triangles ABC, AB_1C_1, AB_2C_2, AB_3C_3 are equiangular, hence similar. Region $B_3C_3C_2B_2$ has area 225.



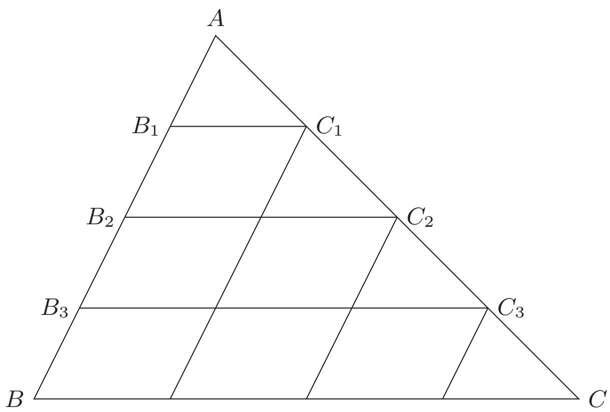
$\boxed{1}$

Since the lines divide AB into four equal segments, the sides and altitudes of the triangles are in the ratio 1:2:3:4. So their areas are in the ratio 1:4:9:16.

Let the area of triangle AB_1C_1 be x. Then $225 = |AB_3C_3| - |AB_2C_2| = 9x - 4x = 5x$ and the area of triangle ABC is $16x = 16 \times \frac{225}{5} = 16 \times 45 = \mathbf{720}$. $\boxed{1}$

Method 2

Let B_1C_1, B_2C_2, B_3C_3, be the lines parallel to BC. Draw lines parallel to AB as shown. This produces 4 small congruent triangles and 6 small congruent parallelograms.



$\boxed{1}$

Drawing the diagonal from top left to bottom right in any parallelogram produces two triangles that are congruent to the top triangle. Thus triangle ABC can be divided into 16 congruent triangles. The region $B_3C_3C_2B_2$ has area 225 and consists of 5 of these triangles. Hence $225 = \frac{5}{16} \times |ABC|$ and $|ABC| = \frac{16}{5} \times 225 = \mathbf{720}$. $\boxed{1}$

3. *Method 1*

Firstly, we note that the combined area of the 12 student cards is
$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + 121 + 144 = 650$. $\boxed{1}$
(Alternatively, use $1 + 2^3 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$.)

According to Alice and Bob, $650 - x^2 = y^2$ for some integers x and y, where $1 \le x \le 12$.
So $y^2 \ge 650 - 144 = 506$ and $y^2 \le 650 - 1 = 649$. Therefore $23 \le y \le 25$. $\boxed{1}$

If $y = 23$, then $x = 11$. If $y = 24$, then x is not an integer. If $y = 25$, then $x = 5$.

Thus $a = 5$ and $b = 11$ or vice versa. So $ab = 5 \times 11 = \mathbf{55}$. $\boxed{1}$

Method 2

Firstly, we note that the combined area of the 12 student cards is
$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + 121 + 144 = 650$. $\boxed{1}$
(Alternatively, use $1 + 2^3 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$.)

According to Alice, $650 - a^2 = c^2$ for some integer c. Since 650 is even, a and c must both be even or odd. If a and c are even, then a^2 and c^2 are multiples of 4. But 650 is not a multiple of 4, so a and c are odd. $\boxed{1}$

We try odd values for a from 1 to 11.

$650 - 1^2 = 649$, which is not a perfect square.
$650 - 3^2 = 641$, which is not a perfect square.
$650 - 5^2 = 625$, which is 25^2, giving one of the solutions.
$650 - 7^2 = 601$, which is not a perfect square.
$650 - 9^2 = 569$, which is not a perfect square.
$650 - 11^2 = 529$, which is 23^2, giving the second solution.

Thus $a = 5$ and $b = 11$ or vice versa. So $ab = 5 \times 11 = \mathbf{55}$. $\boxed{1}$

4. *Method 1*

Let the value of the three types of coin be a, b, c and let Jack's collection be $2a+b+c=28$. Then, swapping b with c if necessary, Jill's collection is one of:

$$3a+b+c, \quad 2a+2b+c, \quad a+2b+2c, \quad a+3b+c. \qquad \boxed{1}$$

Since $3a+b+c$ and $2a+2b+c$ are greater than 28, Jill's collection is either $a+2b+2c$ or $a+3b+c$. If $a+2b+2c=21$, then adding $2a+b+c=28$ gives $3(a+b+c)=49$, which is impossible since 3 is not a factor of 49. $\boxed{1}$

So $a+3b+c=21$. Subtracting from $2a+b+c=28$ gives $a=2b+7$, which means a is odd and at least 9. If $a=9$, then $b=1$ and $c=9$. But a, b, c must be distinct, so a is at least 11. Since $b+c \geq 3$, we have $2a \leq 25$ and $a \leq 12$. Hence $a=11$, $b=2$, $c=4$ and $a+b+c=\mathbf{17}$. $\boxed{1}$

Method 2

Let the value of the three types of coin be a, b, c. Then Jill's collection is one of:

$$2a+2b+c, \quad 3a+b+c.$$

And Jack's collection is one of:

$$2a+b+c, \quad a+2b+c, \quad a+b+2c. \qquad \boxed{1}$$

Suppose Jill's collection is $2a+2b+c=21$. Since $2a+b+c$ and $a+2b+c$ are less than $2a+2b+c$, Jack's collection must be $a+b+2c=28$. Adding this to Jill's yields $3(a+b+c)=49$, which is impossible since 3 is not a factor of 49. $\boxed{1}$

So Jill's collection is $3a+b+c=21$. Since $2a+b+c$ is less than $3a+b+c$, Jack's collection must be $a+2b+c=28$ or $a+b+2c=28$. Swapping b with c if necessary, we may assume that $a+2b+c=28$. Subtracting $3a+b+c=21$ gives $b=2a+7$ and $c=14-5a$. So $a \leq 2$. If $a=1$, then $b=9=c$. Hence $a=2$, $b=11$, $c=4$ and $a+b+c=\mathbf{17}$. $\boxed{1}$

Method 3

Let the value of the three types of coin be a, b, c, where $1 \leq a < b < c$. Then Jack's collection is one of:

$$2a+b+c, \quad a+2b+c, \quad a+b+2c.$$

And Jills' collection is one of:

$$3a+b+c, \quad 2a+2b+c, \quad 2a+b+2c, \quad a+3b+c, \quad a+2b+2c, \quad a+b+3c.$$

All of Jill's possible collections exceed $2a+b+c$, so Jack's collection is $a+2b+c$ or $a+b+2c$. All of Jill's possible collections exceed $a+2b+c$, except possibly for $3a+b+c$. If $3a+b+c=21$, then subtracting from $a+2b+c=28$ gives $b=7+2a \geq 9$. But then $a+2b+c \geq 1+18+10 > 28$. $\boxed{1}$

So Jack's collection is $a+b+2c=28$. Then $a+b$ is even, hence $b \geq 3$, $a+b \geq 4$, $2c=28-a-b \leq 24$, and $c \leq 12$. Of Jill's possible collections, only $3a+b+c$, $2a+2b+c$, and $a+3b+c$ could be less than $a+b+2c$. If $a+3b+c=21$, then subtracting from $a+b+2c=28$ gives $c=7+2b$, which means $c \geq 13$. If $2a+2b+c=21$, then subtracting from $2a+2b+4c=56$ gives $3c=35$, which means c is a fraction. $\boxed{1}$

So $3a+b+c=21$. Subtracting from $a+b+2c=28$ gives $c=7+2a$, which means c is odd and at least 9. If $c=9$, then $a=1$ and $b=9=c$. So $c=11$, $a=2$, $b=4$ and $a+b+c=\mathbf{17}$. $\boxed{1}$

Method 4

Let the value of the three types of coin be a, b, c, where $1 \le a < b < c$.

Then Jack's collection is $28 = a + b + c + d$ where d equals one of a, b, c. Since $a + b \ge 3$, $c + d \le 25$. So $d \le 25 - c \le 25 - d$. Then $2d \le 25$, hence $d \le 12$, which implies $a + b + c \ge 16$.

Jills' collection is $21 = a+b+c+e$ where e is the sum of two of a, b, c with repetition permitted. So $e \ge 2a \ge 2$. Hence $a + b + c \le 19$. $\boxed{1}$

From $a + b + c + d = 28$ and $16 \le a + b + c \le 19$, we get $9 \le d \le 12$.
If $d = a$, then $a + b + c + d > 4d \ge 36$. If $d = b$, then $a + b + c + d > 1 + 3d \ge 28$. So $d = c$.

From $21 = a + b + c + e \ge 16 + e \ge 16 + 2a$ we get $2a \le 5$, hence $a \le 2$. $\boxed{1}$

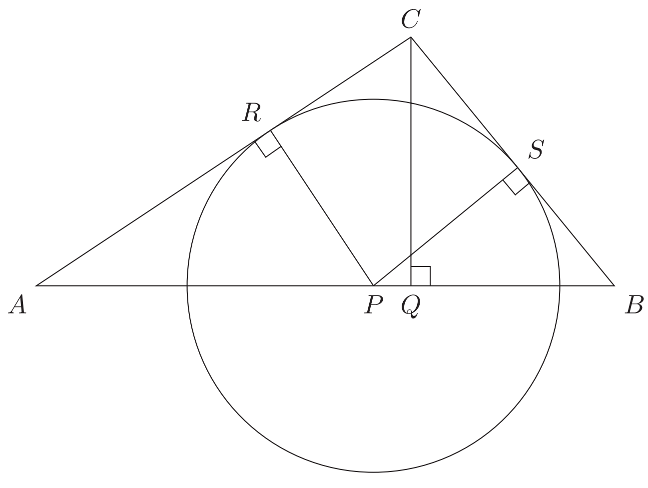
The following table lists all cases. Note that each of x and y equals one of a, b, c.

a	$a+b+c$	d	c	b	e	comment
1	16	12	12	3	5	$e \ne x + y$
1	17	11	11	5	4	$e \ne x + y$
1	18	10	10	7	3	$e \ne x + y$
1	19	9	9	9	2	$b = c$
2	16	12	12	2	5	$a = b$
2	17	11	11	4	4	$e = 2a$
2	18	10	10	6	3	$e \ne x + y$
2	19	9	9	8	2	$e \ne x + y$

So $a = 2$, $b = 4$, $c = 11$, $d = 11$, $e = 4$, and $a + b + c = \mathbf{17}$. $\boxed{1}$

5. *Method 1*

Let CA touch the circle at R and CB touch the circle at S. Let Q be a point on AB so that CQ and AB are perpendicular.



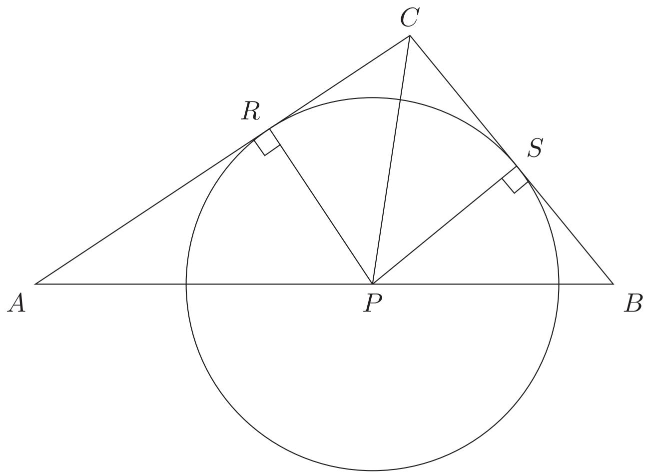
$\boxed{1}$

Let r be the radius of the circle.
From similar triangles AQC and ARP, $CQ/r = 70/AP$.
From similar triangles BQC and BSP, $CQ/r = 50/BP = 50/(90 - AP)$. $\boxed{1}$

Hence $7(90 - AP) = 5AP$, $630 = 12AP$, $2AP = \mathbf{105}$. $\boxed{1}$

Method 2

Let CA touch the circle at R and CB touch the circle at S.



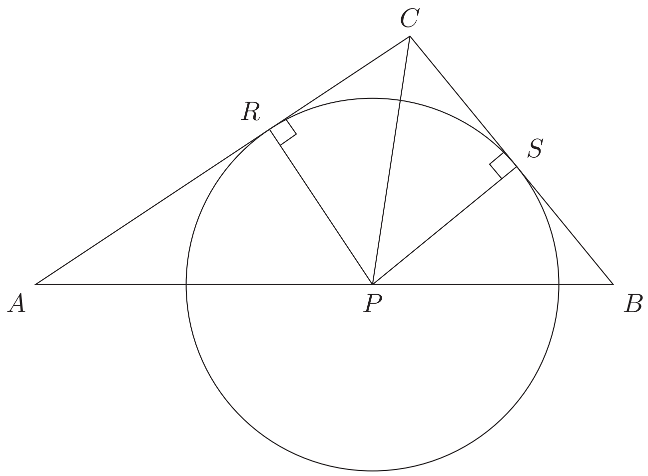
$\boxed{1}$

The radius of the circle is the height of triangle APC on base AC and the height of triangle BPC on base BC. So ratio of the area of APC to the area of BPC is $AC : PC = 7 : 5$. $\boxed{1}$

Triangles APC and BPC also have the same height on bases AP and BP. So the ratio of their areas is $AP : (90 - AP)$. Hence $5AP = 7(90 - AP)$, $12AP = 630$, and $2AP = \mathbf{105}$. $\boxed{1}$

Method 3

Let CA touch the circle at R and CB touch the circle at S.



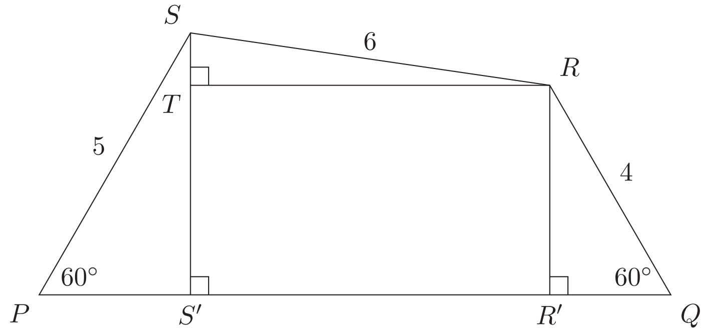
$\boxed{1}$

Since $PR = PS$, right-angled triangles PRC and PSC are congruent. Hence CP bisects $\angle ACB$. $\boxed{1}$

From the angle bisector theorem, $AP/PB = AC/BC = 7/5$. Hence $5AP = 7(90 - AP)$, $12AP = 630$, and $2AP = \mathbf{105}$. $\boxed{1}$

6. *Method 1*

Let SS' and RR' be perpendicular to PQ with S' and R' on PQ. Let RT be perpendicular to SS' with T on SS'.



$\boxed{1}$

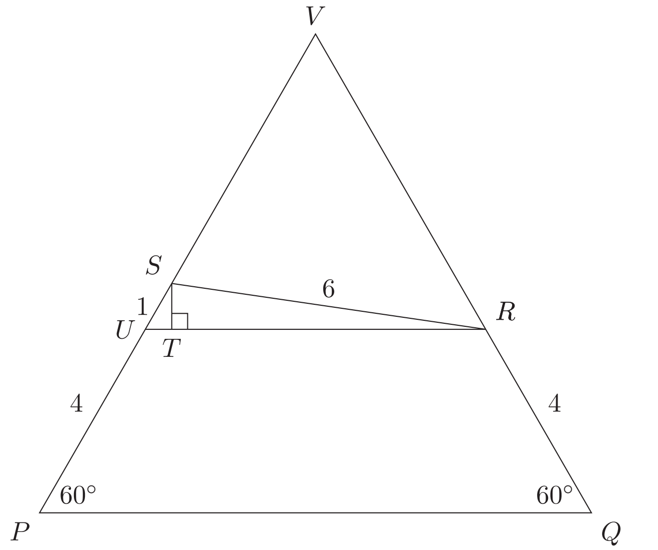
Since $\angle P = 60°$, $PS' = 5/2$ and $SS' = 5\sqrt{3}/2$.
Since $\angle Q = 60°$, $QR' = 2$ and $RR' = 2\sqrt{3}$. $\boxed{1}$

Hence $ST = SS' - TS' = SS' - RR' = 5\sqrt{3}/2 - 2\sqrt{3} = \sqrt{3}/2$.
Applying Pythagoras' theorem to $\triangle RTS$ gives $RT^2 = 36 - \frac{3}{4} = 141/4$. $\boxed{1}$

So $PQ = PS' + S'R' + R'Q = PS' + TR + R'Q = 5/2 + \sqrt{141}/2 + 2$.
Hence $a + \sqrt{b} = 2PQ = 9 + \sqrt{141}$. An obvious solution is $a = 9$, $b = 141$.
Given that a and b are unique, we have $a + b = \mathbf{150}$. $\boxed{1}$

Method 2

Let U be the point on PS so that UR is parallel to PQ. Let T be the point on RU so that ST is perpendicular to RU. Extend PS and QR to meet at V.



$\boxed{1}$

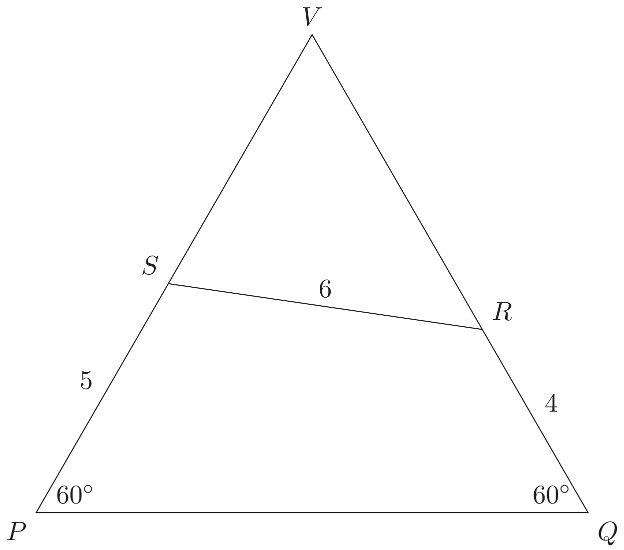
Triangle PQV is equilateral. Since $UR \parallel PQ$, $\triangle URV$ is equilateral and $PU = QR = 4$.
So $US = 1$, $UT = \frac{1}{2}$, $ST = \frac{\sqrt{3}}{2}$. $\boxed{1}$

Applying Pythagoras' theorem to $\triangle RTS$ gives $RT^2 = 36 - \frac{3}{4} = 141/4$. $\boxed{1}$

We also have $RT = RU - UT = RV - \frac{1}{2} = QV - \frac{9}{2} = PQ - \frac{9}{2}$.
So $2PQ = 9 + \sqrt{141} = a + \sqrt{b}$. Given that a and b are unique, we have $a + b = \mathbf{150}$. $\boxed{1}$

Method 3

Extend PS and QR to meet at V.



$\boxed{1}$

Triangle PQV is equilateral. Let $PQ = x$. Then $VS = x - 5$ and $VR = x - 4$. $\boxed{1}$

Applying the cosine rule to $\triangle RVS$ gives

$$\begin{aligned} 36 &= (x-4)^2 + (x-5)^2 - 2(x-4)(x-5)\cos 60^\circ \\ &= (x^2 - 8x + 16) + (x^2 - 10x + 25) - (x^2 - 9x + 20) \\ 0 &= x^2 - 9x - 15 \end{aligned} \qquad \boxed{1}$$

Hence $2x = 9 + \sqrt{81 + 60} = a + \sqrt{b}$. Given that a and b are unique, we have $a + b = \mathbf{150}$. $\boxed{1}$

Comment

We can prove that a and b are unique as follows. We have $(a-9)^2 = 141 + b - 2\sqrt{141b}$. So $2\sqrt{141b}$ is an integer, hence $141b$ is a perfect square. Since $141 = 3 \times 47$ and 3 and 47 are prime, $b = 141m^2$ for some integer m. Hence $|a - 9| = \sqrt{141}|m - 1|$. If neither side of this equation is 0, then we can rewrite it as $r = \sqrt{141}s$ where r and s are coprime integers, giving $r^2 = 141s^2 = 3 \times 47 \times s^2$. So 3 divides r^2. Then 3 divides r, 9 divides r^2, 9 divides $3s^2$, 3 divides s^2, hence 3 divides s, a contradiction. So both sides of the equation are 0. Therefore $a = 9$ and $b = 141$.

7. *Method 1*

Let there be r red sweets and g green sweets. We may assume $r \geq 2$. If Dan puts the first sweet back, then the probability that the two selected sweets are red is

$$\frac{r}{r+g} \times \frac{r}{r+g}. \qquad \boxed{1}$$

If Dan eats the first sweet, then the probability that the two selected sweets are red is

$$\frac{r}{r+g} \times \frac{r-1}{r+g-1}. \qquad \boxed{1}$$

The first probability is 105% of the second, so dividing and rearranging gives

$$\begin{aligned} \frac{r}{r+g} \times \frac{r+g-1}{r-1} &= \frac{105}{100} = \frac{21}{20} \\ 20\left(\frac{r+g-1}{r+g}\right) &= 21\left(\frac{r-1}{r}\right) \\ 20\left(1 - \frac{1}{r+g}\right) &= 21\left(1 - \frac{1}{r}\right) \\ \frac{21}{r} &= 1 + \frac{20}{r+g} > 1 \end{aligned} \qquad \boxed{1}$$

So $r < 21$. If $r = 20$, then $\frac{1}{20} = \frac{20}{r+g}$, and $r + g = 400$.
If $r + g$ increases, then $1 + \frac{20}{r+g}$ and therefore $\frac{21}{r}$ decrease, so r increases.
Since r cannot exceed 20, $r + g$ cannot exceed 400.
So the largest number of sweets in the jar is **400**. $\boxed{1}$

Method 2

Let there be r red sweets and g green sweets. We may assume $r \geq 2$. If Dan puts the first sweet back, then the probability that the two selected sweets are red is

$$\frac{r}{r+g} \times \frac{r}{r+g}. \qquad \boxed{1}$$

If Dan eats the first sweet, then the probability that the two selected sweets are red is

$$\frac{r}{r+g} \times \frac{r-1}{r+g-1}. \qquad \boxed{1}$$

The first probability is 105% of the second, so dividing and rearranging gives

$$\begin{aligned} \frac{r}{r+g} \times \frac{r+g-1}{r-1} &= \frac{105}{100} = \frac{21}{20} \\ 20r(r+g-1) &= 21(r+g)(r-1) \\ 20r(r+g) - 20r &= 21r(r+g) - 21(r+g) \\ r + 21g &= r(r+g) \\ r + g &= 1 + 21g/r \end{aligned} \qquad \boxed{1}$$

If $r \geq 21$, then $r + g \geq 21 + g$ and $1 + 21g/r \leq 1 + g$, a contradiction. So $r \leq 20$.

If $r = 20$, then $20 + g = 1 + 21g/20$, hence $g = 400 - 20 = 380$ and $r + g = 400$.

We also have the equation $(21 - r)g = r(r - 1)$.
If $r < 20$, then $g < (21 - r)g = r(r - 1) < 20 \times 19 = 380$, hence $r + g < 20 + 380 = 400$.
So the largest number of sweets in the jar is **400**. $\boxed{1}$

Method 3

Let there be r red sweets and g green sweets. We may assume $r \geq 2$. Let $n = r + g$. Then the probability of selecting two red sweets if the first sweet is put back is

$$\frac{r}{n} \times \frac{r}{n} \qquad \boxed{1}$$

and the probability if Dan eats the first sweet before selecting the second is

$$\frac{r}{n} \times \frac{r-1}{n-1}. \qquad \boxed{1}$$

The first probability is 105% of the second, so dividing and rearranging gives

$$\begin{aligned} \frac{r}{n} \times \frac{n-1}{r-1} &= \frac{105}{100} = \frac{21}{20} \\ 20r(n-1) &= 21n(r-1) \\ 21n - nr - 20r &= 0 \\ (n+20)(21-r) &= 420 \qquad \boxed{1} \end{aligned}$$

Since $n + 20$ is positive, $21 - r$ is positive.
Hence n is largest when $21 - r = 1$ and then $n + 20 = 420$.
So the largest number of sweets in the jar is **400**. $\boxed{1}$

Comment

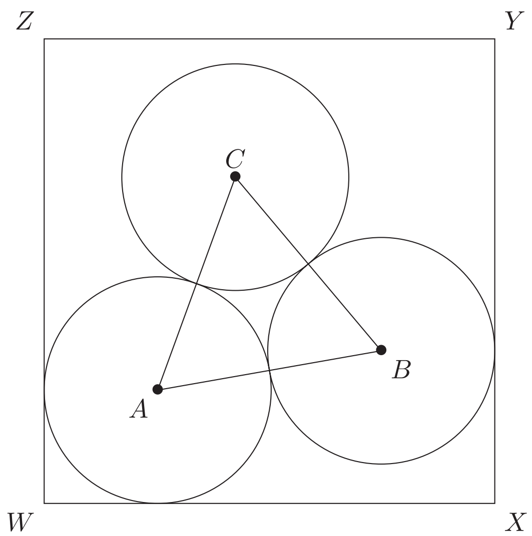
Since $21 - r$ is a factor of 420 and $2 \leq r \leq 20$, the following table gives all possible values of r, n, g.

$21-r$	$n+20$	r	n	g
1	420	20	400	380
2	210	19	190	171
3	140	18	120	102
4	105	17	85	68
5	84	16	64	48
6	70	15	50	35
7	60	14	40	26
10	42	11	22	11
12	35	9	15	6
14	30	7	10	3
15	28	6	8	2

.

8. Let $WXYZ$ be a square that encloses the three circles and is as small as possible. Let the centres of the three given circles be A, B, C. Then ABC is an equilateral triangle of side length 1. We may assume that A, B, C are arranged anticlockwise and that the circle with centre A touches WX and WZ. We may also assume that WX is horizontal.

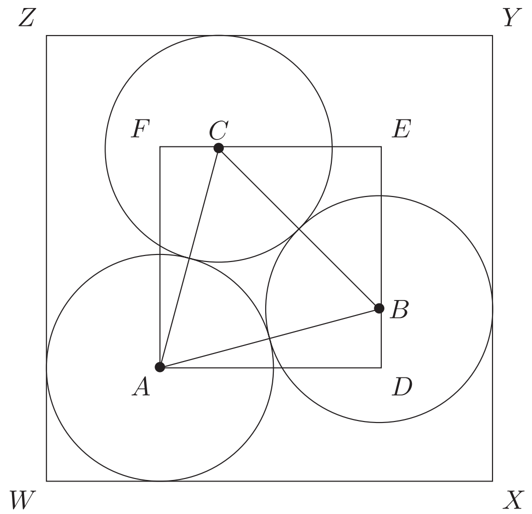
Note that if neither YX nor YZ touch a circle, then the square can be contracted by moving Y along the diagonal WY towards W. So at least one of YX and YZ must touch a circle and it can't be the circle with centre A. We may assume that XY touches the circle with centre B.



If YZ does not touch a circle, then the 3-circle cluster can be rotated anticlockwise about A allowing neither YX nor YZ to touch a circle. So YZ touches the circle with centre C. $\boxed{1}$

Method 1

Let $ADEF$ be the rectangle with sides through C and B parallel to WX and WZ respectively.



$\boxed{1}$

Since $AF = WZ - 1 = WX - 1 = AD$, $ADEF$ is a square.
Since $AC = 1 = AB$, triangles AFC and ADB are congruent. So $FC = DB$ and $CE = BE$.
Let $x = AD$. Since $AB = 1$ and triangle ADB is right-angled, $DB = \sqrt{1 - x^2}$.

Since CBE is right-angled isosceles with $BC = 1$, we have $BE = 1/\sqrt{2}$.
So $x = DE = \sqrt{1 - x^2} + 1/\sqrt{2}$. $\boxed{1}$

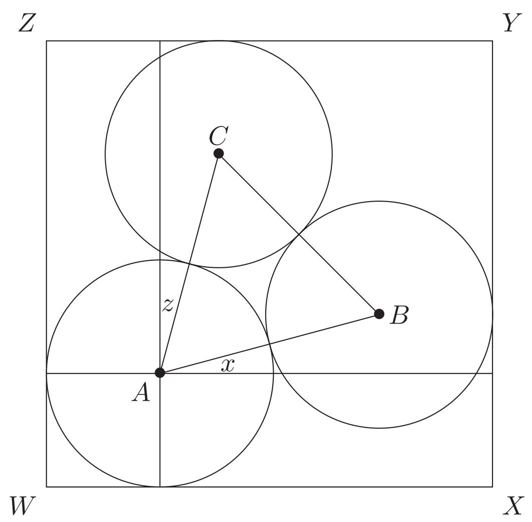
Squaring both sides of $x - 1/\sqrt{2} = \sqrt{1 - x^2}$ gives

$$\begin{aligned} 1 - x^2 &= (x - 1/\sqrt{2})^2 = x^2 - \sqrt{2}x + 1/2 \\ 0 &= 2x^2 - \sqrt{2}x - 1/2 \\ x &= (\sqrt{2} \pm \sqrt{2+4})/4 \end{aligned}$$

Since $x > 0$, we have $x = (\sqrt{2} + \sqrt{6})/4 = (\sqrt{18} + \sqrt{54})/12$. Hence $WX = 1 + (\sqrt{18} + \sqrt{54})/12$. We are told that $WX = a + (\sqrt{b} + \sqrt{c})/12$ where a, b, c are unique integers. This gives $a + b + c = 1 + 18 + 54 = \mathbf{73}$. $\boxed{1}$

Method 2

Draw lines through A parallel to WX and WZ.



$\boxed{1}$

With angles x and z as shown, we have

$$\begin{aligned} WX &= \frac{1}{2} + AB\cos x + \frac{1}{2} = 1 + \cos x \\ WZ &= \frac{1}{2} + AC\cos z + \frac{1}{2} = 1 + \cos z \end{aligned}$$

Since $WX = WZ$, $x = z$. Since $x + 60° + z = 90°$, we have $x = 15°$. So $\boxed{1}$

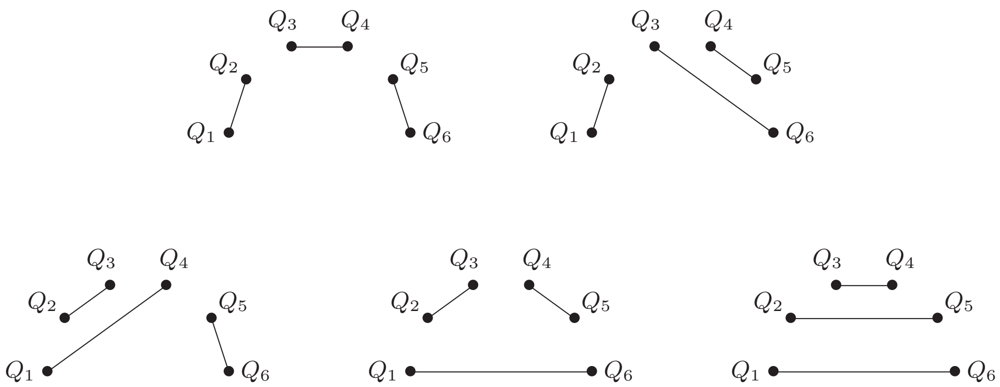
$$\begin{aligned} WX &= 1 + \cos 15° = 1 + \cos(45° - 30°) \\ &= 1 + \cos 45° \cos 30° + \sin 45° \sin 30° \\ &= 1 + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} \\ &= 1 + \frac{1+\sqrt{3}}{2\sqrt{2}} = 1 + \frac{\sqrt{2}+\sqrt{6}}{4} = 1 + \frac{\sqrt{18}+\sqrt{54}}{12} \end{aligned}$$

We are told that $WX = a + (\sqrt{b} + \sqrt{c})/12$ where a, b, c are unique integers. This gives $a + b + c = 1 + 18 + 54 = \mathbf{73}$. $\boxed{1}$

9. *Method 1*

Let the pair of intersecting lines be AC and BD where A, B, C, D are four of the ten given points. These lines split the remaining six points into four subsets S_1, S_2, S_3, S_4. For each i, each line segment beginning in S_i also ends in S_i, otherwise AC and BD would not be the only intersecting pair of lines. Thus each S_i contains an even number of points, from 0 to 6. $\boxed{1}$

If S_i contains 2 points, then there it has only 1 line segment. If S_i contains 4 points, then there are precisely 2 ways to connect its points in pairs by non-crossing segments. If S_i contains 6 points, let the points be $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ in clockwise order. To avoid crossing segments, Q_1 must be connected to one of Q_2, Q_4, Q_6. So, as shown, there are precisely 5 ways to connect the six points in pairs by non-crossing segments.



$\boxed{1}$

In some order, the sizes of S_1, S_2, S_3, S_4 are $\{6, 0, 0, 0\}$, $\{4, 2, 0, 0\}$, or $\{2, 2, 2, 0\}$. We consider the three cases separately.

In the first case, by rotation about the circle, there are 10 ways to place the S_i that has 6 points. Then there are 5 ways to arrange the line segments within that S_i. So the number of ways to draw the line segments in this case is $10 \times 5 = 50$. $\boxed{1}$

In the second case, in clockwise order, the sizes of the S_i must be (4, 2, 0, 0), (4, 0, 2, 0) or (4, 0, 0, 2). In each case, by rotation about the circle, there are 10 ways to place the S_i. Then there are 2 ways to arrange line segments within the S_i that has 4 points, and there is 1 way to arrange the line segment within the S_i that has 2 points. So the number of ways to draw the line segments in this case is $3 \times 10 \times 2 \times 1 = 60$. $\boxed{1}$

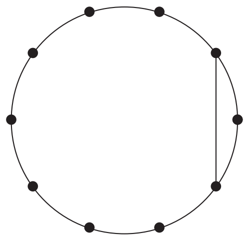
In the third case, in clockwise order, the sizes of the S_i must be (2, 2, 2, 0). By rotation about the circle, there are 10 ways to place the S_i. Then there is only 1 way to arrange the line segment within each S_i that has 2 points. So there are 10 ways to arrange the line segments in this case.

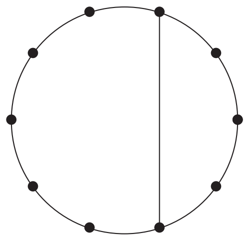
In total, the number of ways to arrange the line segments is $50 + 60 + 10 = \mathbf{120}$. $\boxed{1}$

Method 2

The pair of intersecting lines partition the circle into four arcs. In order to allow the remaining points to be paired up without further crossings, we require each such arc to contain an even number of points. $\boxed{1}$

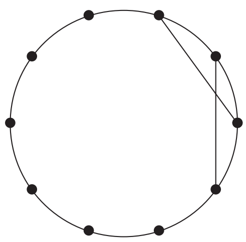
So each line of a crossing pair partitions the circle into two arcs, each of which contain an odd number of points. Disregarding rotation of the circle, a crossing line is one of only two types.





$\boxed{1}$

So, disregarding rotations, there are only four ways to have the pair of crossing lines. Underneath each diagram we list the number of ways of joining up the remaining pairs of points without introducing more crossings. (The number 5 is justified in Method 1.)



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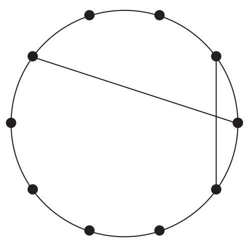

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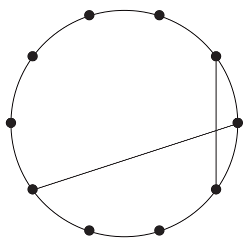

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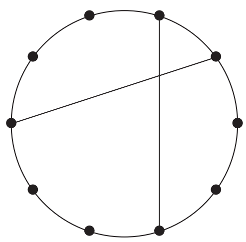
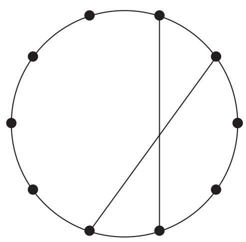


1



2

$\boxed{2}$

So, counting rotations, the number of pairings with a single crossing is
$10 \times (5 + 2 + 2 + 1 + 2) = \mathbf{120}$. $\boxed{1}$

10. Since the 2nd, 4th, 6th, 8th and 10th digits must be even, the other digits must be odd. Since the last digit must be 0, the fifth digit must be 5. $\boxed{1}$

Let a be the 3rd digit and b be the 4th digit. If b is 4 or 8, then 4 divides b but does not divide $10a$ since a is odd. Hence 4 does not divide $10a + b$. So the 4th digit is 2 or 6.

Now let a, b, c be the 6th, 7th, 8th digits respectively. If c is 8, then 8 divides $100a + c$ but does not divide $10b$ since b is odd. Hence 8 does not divide $100a + 10b + c$. If c is 4, then 8 divides $100a$ but does not divide $10b + c = 2(5b + 2)$ since b is odd. Hence 8 does not divide $100a + 10b + c$. So the 8th digit is 2 or 6.

So each of the 2nd and 6th digits is 4 or 8. $\boxed{1}$

Since 3 divides the sum of the first three digits and the sum of the first six digits, it also divides the sum of the 4th, 5th, and 6th digits. So the 4th, 5th, and 6th digits are respectively 2 5 8 or 6 5 4. Thus we have two cases with a, b, c, d equal to 1, 3, 7, 9 in some order. $\boxed{1}$

Case 1. a 4 b 2 5 8 c 6 d 0

Since 3 divides $a + 4 + b$, one of a and b equals 1 and the other is 7. Since 8 divides 8 c 6, c is 9. So we have 1 4 7 2 5 8 9 6 d 0 or 7 4 1 2 5 8 9 6 d 0. But neither 1 4 7 2 5 8 9 nor 7 4 1 2 5 8 9 is a multiple of 7. $\boxed{1}$

Case 2. a 8 b 6 5 4 c 2 d 0

Since 8 divides 4 c 2, c is 3 or 7.

If $c = 3$, then, because 3 divides $a + 8 + b$, we have one of:

1 8 9 6 5 4 3 2 d 0, 7 8 9 6 5 4 3 2 d 0, 9 8 1 6 5 4 3 2 d 0, 9 8 7 6 5 4 3 2 d 0.

But none of 1 8 9 6 5 4 3, 7 8 9 6 5 4 3, 9 8 1 6 5 4 3, 9 8 7 6 5 4 3 is a multiple of 7.

If $c = 7$, then, because 3 divides $a + 8 + b$, we have one of:

1 8 3 6 5 4 7 2 d 0, 1 8 9 6 5 4 7 2 d 0, 3 8 1 6 5 4 7 2 d 0, 9 8 1 6 5 4 7 2 d 0.

None of 1 8 3 6 5 4 7, 1 8 9 6 5 4 7, 9 8 1 6 5 4 7 is a multiple of 7.

This leaves 3 8 1 6 5 4 7 2 9 0 as the only draw. $\boxed{1}$

Investigation

Note that B must play an even digit on each turn.

If A starts with 2, then B can only respond with 20, 24, 26, or 28. A may then leave one of 204, 240, 261, 285. B cannot respond to 261. The other numbers force respectively 20485, 24085, 28560. B cannot respond to any of these. $\boxed{\text{bonus 1}}$

If A starts with 4, then B can only respond with 40, 42, 46, or 48. A may then leave one of 408, 420, 462, 480. B cannot respond to 408 and 480. Each of the other numbers force one of 42085, 46205, 46280, 46285. B cannot respond to any of these. $\boxed{\text{bonus 1}}$

If A starts with 6, then B can only respond with 60, 62, 64, or 68. A may then leave one of 609, 621, 648, 684. B cannot respond to 621. The other numbers force respectively 60925, 64805, 68405. B can only respond with 609258. Then A may reply with 6092583, to which B has no response. $\boxed{\text{bonus 1}}$

If A starts with 8, then B can only respond with 80, 82, 84, or 86. A may then leave one of 804, 825, 840, 864. B cannot respond to 804 and 840. The other numbers force respectively 82560, 86405. B cannot respond to either of these. $\boxed{\text{bonus 1}}$

Marking Scheme

1. Establishing a relevant quadratic equation. $\boxed{1}$
Correct answer (57). $\boxed{1}$

2. A valid approach with a relevant diagram. $\boxed{1}$
Correct answer (720). $\boxed{1}$

3. Correct total area of the 12 cards. $\boxed{1}$
A useful equation in two variables and useful constraints on the variables. $\boxed{1}$
Correct answer (55). $\boxed{1}$

4. A valid approach with some progress. $\boxed{1}$
Further progress. $\boxed{1}$
Correct answer (17). $\boxed{1}$

5. A useful diagram. $\boxed{1}$
Establishing useful ratios. $\boxed{1}$
Correct answer (105). $\boxed{1}$

6. A useful diagram. $\boxed{1}$
Some useful calculations. $\boxed{1}$
More calculations. $\boxed{1}$
Correct answer (150). $\boxed{1}$

7. Establishing a relevant probability expression. $\boxed{1}$
Establishing another relevant probability expression. $\boxed{1}$
Establishing a simplified equation. $\boxed{1}$
Correct answer (400). $\boxed{1}$

8. Establishing 4 square sides touch a circle. $\boxed{1}$
A useful construction. $\boxed{1}$
Establishing a useful equation in one variable. $\boxed{1}$
Correct answer (73). $\boxed{1}$

9. Useful approach. $\boxed{1}$
Some progress. $\boxed{1}$
Establishing relevant cases and correctly enumerating one case. $\boxed{1}$
Correctly enumerating another case. $\boxed{1}$
Completing the enumeration with the correct answer (120). $\boxed{1}$

10. Establishing places for odd digits and 0 and 5. $\boxed{1}$
Establishing relevant options for the even digits. $\boxed{1}$
Establishing relevant cases. $\boxed{1}$
Correctly resolving one case. $\boxed{1}$
Completing the proof with the correct answer (3816547290). $\boxed{1}$

Investigation:

Proving A wins with first digit 2. $\boxed{\text{bonus 1}}$
Proving A wins with first digit 4. $\boxed{\text{bonus 1}}$
Proving A wins with first digit 6. $\boxed{\text{bonus 1}}$
Proving A wins with first digit 8. $\boxed{\text{bonus 1}}$