ĐỀ THI VÀ ĐÁP ÁN CHI TIẾT

CUỘC THI VÔ ĐỊCH TOÁN CẤP TRUNG HỌC ÚC MỞ RỘNG NĂM 2015

2015 Australian Intermediate Mathematics Olympiad - Questions

Time allowed: 4 hours. NO calculators are to be used.

Questions 1 to 8 only require their numerical answers all of which are non-negative integers less than 1000.
Questions 9 and 10 require written solutions which may include proofs.
The bonus marks for the Investigation in Question 10 may be used to determine prize winners.

1. A number written in base a is 123_a. The same number written in base b is 146_b. What is the minimum value of $a+b$? [**2 marks**]

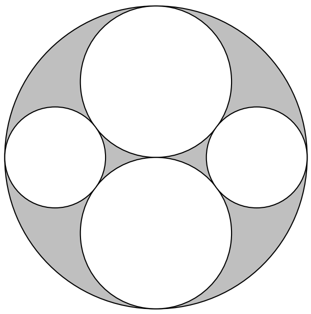
2. A circle is inscribed in a hexagon $ABCDEF$ so that each side of the hexagon is tangent to the circle. Find the perimeter of the hexagon if $AB = 6$, $CD = 7$, and $EF = 8$. [**2 marks**]

3. A selection of 3 whatsits, 7 doovers and 1 thingy cost a total of \$329. A selection of 4 whatsits, 10 doovers and 1 thingy cost a total of \$441. What is the total cost, in dollars, of 1 whatsit, 1 doover and 1 thingy? [**3 marks**]

4. A fraction, expressed in its lowest terms $\frac{a}{b}$, can also be written in the form $\frac{2}{n} + \frac{1}{n^2}$, where n is a positive integer. If $a + b = 1024$, what is the value of a? [**3 marks**]

5. Determine the smallest positive integer y for which there is a positive integer x satisfying the equation $2^{13} + 2^{10} + 2^x = y^2$. [**3 marks**]

6. The large circle has radius $30/\sqrt{\pi}$. Two circles with diameter $30/\sqrt{\pi}$ lie inside the large circle. Two more circles lie inside the large circle so that the five circles touch each other as shown. Find the shaded area.



[**4 marks**]

7. Consider a shortest path along the edges of a 7×7 square grid from its bottom-left vertex to its top-right vertex. How many such paths have no edge above the grid diagonal that joins these vertices? [**4 marks**]

8. Determine the number of non-negative integers x that satisfy the equation

$$\left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor.$$

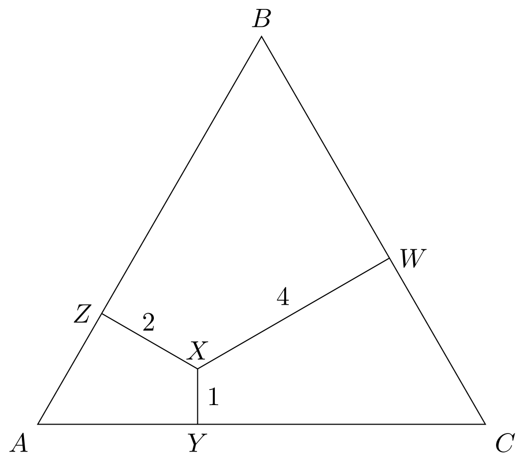
(Note: if r is any real number, then $\lfloor r \rfloor$ denotes the largest integer less than or equal to r.) [**4 marks**]

9. A sequence is formed by the following rules: $s_1 = a, s_2 = b$ and $s_{n+2} = s_{n+1} + (-1)^n s_n$ for all $n \geq 1$.

If $a = 3$ and b is an integer less than 1000, what is the largest value of b for which 2015 is a member of the sequence? Justify your answer. [**5 marks**]

10. X is a point inside an equilateral triangle ABC. Y is the foot of the perpendicular from X to AC, Z is the foot of the perpendicular from X to AB, and W is the foot of the perpendicular from X to BC.

The ratio of the distances of X from the three sides of the triangle is $1 : 2 : 4$ as shown in the diagram.



If the area of $AZXY$ is $13\,\text{cm}^2$, find the area of ABC. Justify your answer. [**5 marks**]

Investigation

If $XY : XZ : XW = a : b : c$, find the ratio of the areas of $AZXY$ and ABC. [**2 bonus marks**]

2015 Australian Intermediate Mathematics Olympiad - Solutions

1. *Method 1*

$$\begin{aligned} 123_a = 146_b &\iff a^2 + 2a + 3 = b^2 + 4b + 6 \\ &\iff (a+1)^2 + 2 = (b+2)^2 + 2 \\ &\iff (a+1)^2 = (b+2)^2 \\ &\iff a + 1 = b + 2 \ (a \text{ and } b \text{ are positive}) \\ &\iff a = b + 1 \end{aligned}$$

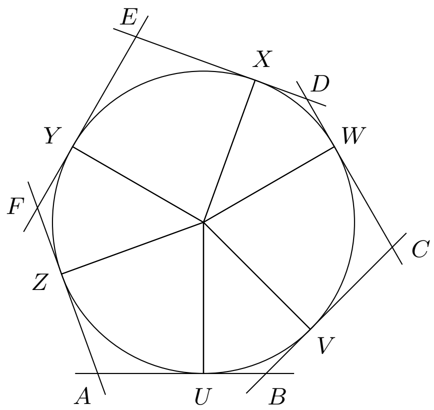
Since the minimum value for b is 7, the minimum value for $a + b$ is $8 + 7 = \mathbf{15}$.

Method 2

Since the digits in any number are less than the base, $b \geq 7$.
We also have $a > b$, otherwise $a^2 + 2a + 3 < b^2 + 4b + 6$.

If $b = 7$ and $a = 8$, then $a^2 + 2a + 3 = 83 = b^2 + 4b + 6$.
So the minimum value for $a + b$ is $8 + 7 = \mathbf{15}$.

2. Let AB, BC, CD, DE, EF, FA touch the circle at U, V, W, X, Y, Z respectively.



Since the two tangents from a point to a circle have equal length,
$UB = BV$, $VC = CW$, $WD = DX$, $XE = EY$, $YF = FZ$, $ZA = AU$.

The perimeter of hexagon $ABCDEF$ is
$AU + UB + BV + VC + CW + WD + DX + XE + EY + YF + FZ + ZA$
$= AU + UB + UB + CW + CW + WD + WD + EY + EY + YF + YF + AU$
$= 2(AU + UB + CW + WD + EY + YF)$
$= 2(AB + CD + EF) = 2(6 + 7 + 8) = 2(21) = \mathbf{42}$.

3. *Preamble*

Let the required cost be x. Then, with obvious notation, we have:

$$\begin{aligned} 3w + 7d + t &= 329 \qquad (1) \\ 4w + 10d + t &= 441 \qquad (2) \\ w + d + t &= x \qquad (3) \end{aligned}$$

Method 1

$3 \times (1) - 2 \times (2)$: $w + d + t = 3 \times 329 - 2 \times 441 = 987 - 882 = \mathbf{105}$.

Method 2

(2) − (1): $w + 3d = 112$.
(1) − (3): $2w + 6d = 329 - x = 2 \times 112 = 224$.

Then $x = 329 - 224 = \mathbf{105}$.

Method 3

$10 \times (1) - 7 \times (2)$: $w = (203 - 3t)/2$
$3 \times (2) - 4 \times (1)$: $d = (7 + t)/2$

Then $w + d + t = 210/2 - 2t/2 + t = \mathbf{105}$.

4. We have $\dfrac{2}{n} + \dfrac{1}{n^2} = \dfrac{2n+1}{n^2}$.

Since $2n + 1$ and n^2 are coprime, $a = 2n + 1$ and $b = n^2$.
So $1024 = a + b = n^2 + 2n + 1 = (n + 1)^2$, hence $n + 1 = 32$.

This gives $a = 2n + 1 = 2 \times 31 + 1 = \mathbf{63}$.

5. *Method 1*

$$\begin{aligned}
2^{13} + 2^{10} + 2^x = y^2 \quad &\iff \quad 2^{10}(2^3 + 1) + 2^x = y^2 \\
&\iff \quad (2^5 \times 3)^2 + 2^x = y^2 \\
&\iff \quad 2^x = y^2 - 96^2 \\
&\iff \quad 2^x = (y + 96)(y - 96).
\end{aligned}$$

Since y is an integer, both $y + 96$ and $y - 96$ must be powers of 2.
Let $y + 96 = 2^m$ and $y - 96 = 2^n$. Then $2^m - 2^n = 192 = 2^6 \times 3$.

Hence $2^{m-6} - 2^{n-6} = 3$. So $2^{m-6} = 4$ and $2^{n-6} = 1$.
In particular, $m = 8$. Hence $y = 2^8 - 96 = 256 - 96 = \mathbf{160}$.

Method 2

We have $y^2 = 2^{13} + 2^{10} + 2^x = 2^{10}(2^3 + 1 + 2^{x-10}) = 2^{10}(9 + 2^{x-10})$.

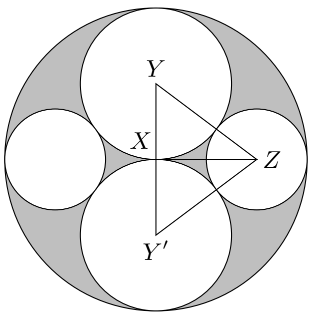
So we want the smallest value of $9 + 2^{x-10}$ that is a perfect square.
Since $9 + 2^{x-10}$ is odd and greater than 9, $9 + 2^{x-10} \geq 25$.

Since $9 + 2^{14-10} = 25$, $y = 2^5 \times 5 = 32 \times 5 = \mathbf{160}$.

Comment

Method 1 shows that $2^{13} + 2^{10} + 2^x = y^2$ has only one solution.

6. The centres Y and Y' of the two medium circles lie on a diameter of the large circle. By symmetry about this diameter, the two smaller circles are congruent. Let X be the centre of the large circle and Z the centre of a small circle.



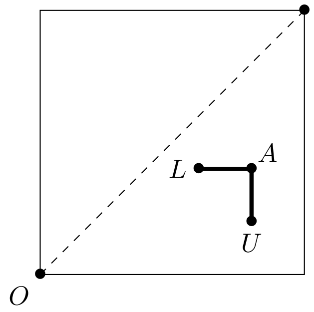
Let R and r be the radii of a medium and small circle respectively. Then $ZY = R + r = ZY'$. Since $XY = XY'$, triangles XYZ and $XY'Z$ are congruent. Hence $XZ \perp XY$.

By Pythagoras, $YZ^2 = YX^2 + XZ^2$. So $(R + r)^2 = R^2 + (2R - r)^2$.
Then $R^2 + 2Rr + r^2 = 5R^2 - 4Rr + r^2$, which simplifies to $3r = 2R$.

So the large circle has area $\pi(30/\sqrt{\pi})^2 = 900$,
each medium circle has area $\pi(15/\sqrt{\pi})^2 = 225$,
and each small circle has area $\pi(10/\sqrt{\pi})^2 = 100$.
Thus the shaded area is $900 - 2 \times 225 - 2 \times 100 = \mathbf{250}$.

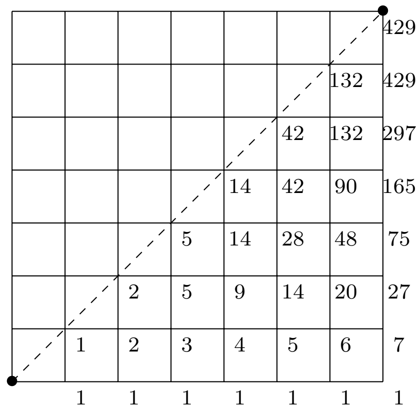
7. *Method 1*

Any path from the start vertex O to a vertex A must pass through either the vertex L left of A or the vertex U underneath A. So the number of paths from O to A is the sum of the number of paths from O to L and the paths from O to U.



There is only one path from O to any vertex on the bottom line of the grid.

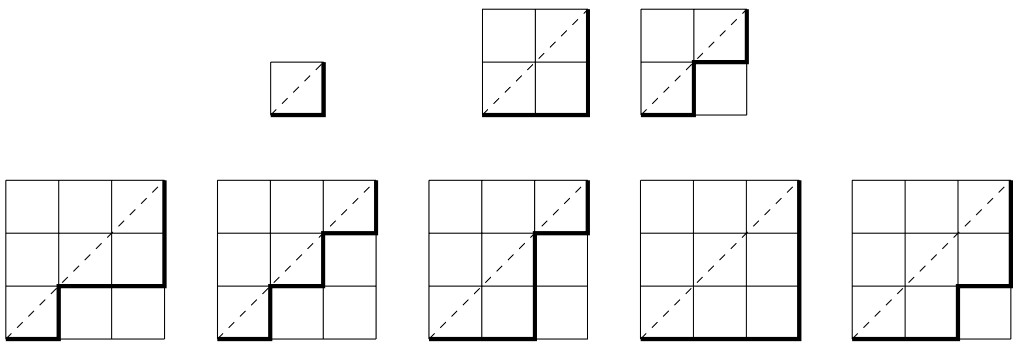
So the number of paths from O to all other vertices can be progressively calculated from the second bottom row upwards as indicated.



Thus the number of required paths is **429**.

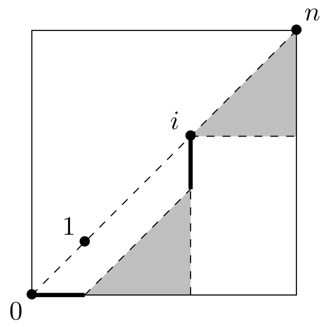
Method 2

To help understand the problem, consider some smaller grids.



Let $p(n)$ equal the number of required paths on an $n \times n$ grid and let $p(0) = 1$.

Starting with the bottom-left vertex, label the vertices of the diagonal 0, 1, 2, ..., n.



Consider all the paths that touch the diagonal at vertex i but not at any of the vertices between vertex 0 and vertex i. Each such path divides into two subpaths.

One subpath is from vertex 0 to vertex i and, except for the first and last edge, lies in the lower triangle of the diagram above. Thus there are $p(i-1)$ of these subpaths.

The other subpath is from vertex i to vertex n and lies in the upper triangle in the diagram above. Thus there are $p(n-i)$ of these subpaths.

So the number of such paths is $p(i-1) \times p(n-i)$.

Summing these products from $i = 1$ to $i = n$ gives all required paths. Thus

$$p(n) = p(n-1) + p(1)p(n-2) + p(2)p(n-3) + \cdots + p(n-2)p(1) + p(n-1)$$

We have $p(1) = 1$, $p(2) = 2$, $p(3) = 5$. So

$p(4) = p(3) + p(1)p(2) + p(3)p(1) + p(3) = 14$,
$p(5) = p(4) + p(1)p(3) + p(2)p(2) + p(3)p(1) + p(4) = 42$,
$p(6) = p(5) + p(1)p(4) + p(2)p(3) + p(3)p(2) + p(1)p(4) + p(5) = 132$, and
$p(7) = p(6) + p(1)p(5) + p(2)p(4) + p(3)p(3) + p(4)p(2) + p(5)p(1) + p(6) = \mathbf{429}$.

8. *Method 1*

Let $\left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor = n$.

Since x is non-negative, n is also non-negative.

If $n = 0$, then x is any integer from 0 to $44 - 1 = 43$: a total of 44 values.

If $n = 1$, then x is any integer from 45 to $2 \times 44 - 1 = 87$: a total of 43 values.

If $n = 2$, then x is any integer from $2 \times 45 = 90$ to $3 \times 44 - 1 = 131$: a total of 42 values.

If $n = k$, then x is any integer from $45k$ to $44(k+1) - 1 = 44k + 43$: a total of $(44k + 43) - (45k - 1) = 44 - k$ values.

Thus, increasing n by 1 decreases the number of values of x by 1. Also the largest value of n is 43, in which case x has only 1 value.

Therefore the number of non-negative integer values of x is $44 + 43 + \cdots + 1 = \frac{1}{2}(44 \times 45) = \mathbf{990}$.

Method 2

Let n be a non-negative integer such that $\left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor = n$.

Then $\left\lfloor \frac{x}{44} \right\rfloor = n \iff 44n \le x < 44(n+1)$ and $\left\lfloor \frac{x}{45} \right\rfloor = n \iff 45n \le x < 45(n+1)$.

So $\left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor = n \iff 45n \le x < 44(n+1) \iff 44n + n \le x < 44n + 44$.

This is the case if and only if $n < 44$, and then x can assume exactly $44 - n$ different values.

Therefore the number of non-negative integer values of x is

$(44 - 0) + (44 - 1) + \cdots + (44 - 43) = 44 + 43 + \cdots + 1 = \frac{1}{2}(44 \times 45) = \mathbf{990}$.

Method 3

Let n be a non-negative integer such that $\left\lfloor \dfrac{x}{44} \right\rfloor = \left\lfloor \dfrac{x}{45} \right\rfloor = n$.

Then $x = 44n + r$ where $0 \le r \le 43$ and $x = 45n + s$ where $0 \le s \le 44$.

So $n = r - s$. Therefore $0 \le n \le 43$. Also $r = n + s$. Therefore $n \le r \le 43$.

Therefore the number of non-negative integer values of x is $44 + 43 + \cdots + 1 = \frac{1}{2}(44 \times 45) = \mathbf{990}$.

9. Working out the first few terms gives us an idea of how the given sequence develops:

n	s_{2n-1}	s_{2n}
1	a	b
2	$b-a$	$2b-a$
3	b	$3b-a$
4	$2b-a$	$5b-2a$
5	$3b-a$	$8b-3a$
6	$5b-2a$	$13b-5a$
7	$8b-3a$	$21b-8a$

It appears that the coefficients in the even terms form a Fibonacci sequence and, from the 5th term, every odd term is a repeat of the third term before it.

These observations are true for the entire sequence since, for $m \ge 1$, we have:

$$\begin{array}{rclcl} s_{2m+2} & = & s_{2m+1} + s_{2m} & & \\ s_{2m+3} & = & s_{2m+2} - s_{2m+1} & = & s_{2m} \\ s_{2m+4} & = & s_{2m+3} + s_{2m+2} & = & s_{2m+2} + s_{2m} \end{array}$$

So, defining $F_1 = 1$, $F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 3$, we have $s_{2n} = bF_n - aF_{n-2}$ for $n \ge 3$. Since $a = 3$ and $b < 1000$, none of the first five terms of the given sequence equal 2015. So we are looking for integer solutions of $bF_n - 3F_{n-2} = 2015$ for $n \ge 3$.

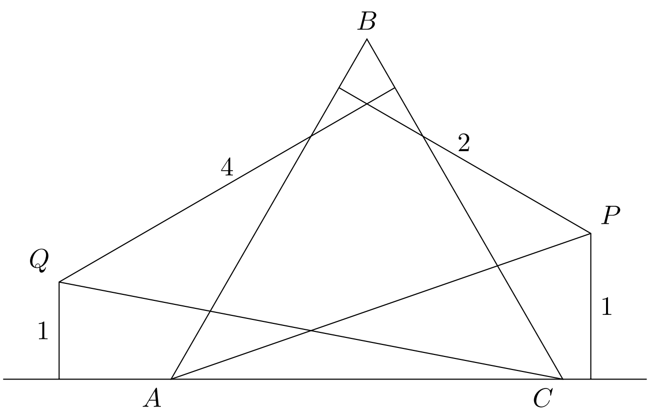
$s_6 = 3b - 3 = 2015$, has no solution.
$s_8 = 5b - 6 = 2015$, has no solution.
$s_{10} = 8b - 9 = 2015$ implies $b = 253$.

For $n \ge 6$ we have $b = 2015/F_n + 3F_{n-2}/F_n$. Since F_n increases, we have $F_n \ge 13$ and $F_{n-2}/F_n < 1$ for $n \ge 6$. Hence $b < 2015/13 + 3 = 158$. So the largest value of b is **253**.

10. *Method 1*

We first show that X is uniquely defined for any given equilateral triangle ABC.

Let P be a point outside $\triangle ABC$ such that its distances from AC and AB are in the ratio 1:2. By similar triangles, any point on the line AP has the same property. Also any point between AP and AC has the distance ratio less than 1:2 and any point between AP and AB has the distance ratio greater than 1:2.

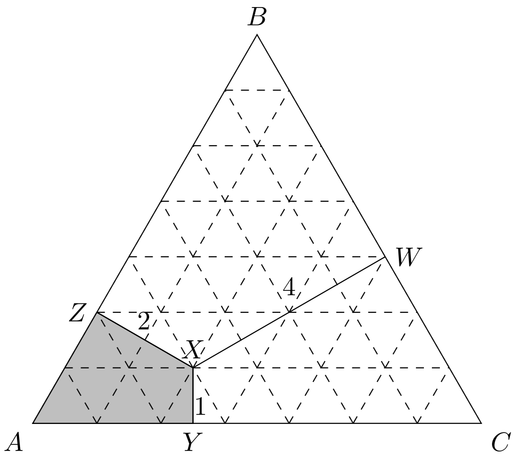


Let Q be a point outside $\triangle ABC$ such that its distances from AC and BC are in the ratio 1:4. By an argument similar to that in the previous paragraph, only the points on CQ have the distance ratio equal to 1:4.

Thus the only point whose distances to AC, AB, and BC are in the ratio 1:2:4 is the point X at which AP and CQ intersect.

Scaling if necessary, we may assume that the actual distances of X to the sides of $\triangle ABC$ are 1, 2, 4. Let h be the height of $\triangle ABC$. Letting $|\ |$ denote area, we have
$|ABC| = \frac{1}{2}h \times AB$ and
$|ABC| = |AXB| + |BXC| + |CXA| = \frac{1}{2}(2AB + 4BC + AC) = \frac{1}{2}AB \times 7.$
So $h = 7$.

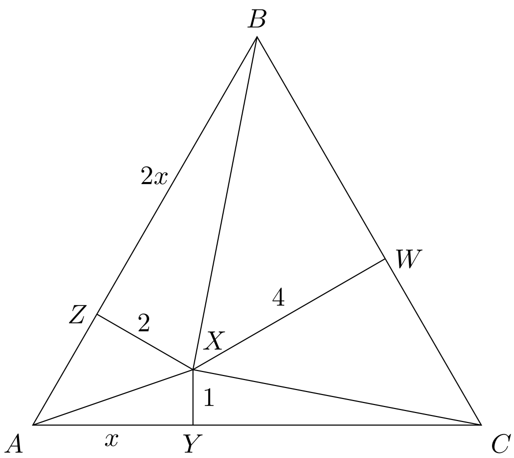
Draw a 7-layer grid of equilateral triangles each of height 1, starting with a single triangle in the top layer, then a trapezium of 3 triangles in the next layer, a trapezium of 5 triangles in the next layer, and so on. The boundary of the combined figure is $\triangle ABC$ and X is one of the grid vertices as shown.



There are 49 small triangles in ABC and 6.5 small triangles in $AZXY$. Hence, after rescaling so that the area of $AZXY$ is $13\,\mathrm{cm}^2$, the area of ABC is $13 \times 49/6.5 = \mathbf{98}\,\mathrm{cm}^2$.

Method 2

Join AX, BX, CX. Since $\angle YAZ = \angle ZBW = 60°$, the quadrilaterals $AZXY$ and $BWXZ$ are similar. Let XY be 1 unit and AY be x. Then $BZ = 2x$.



By Pythagoras: in $\triangle AXY$, $AX = \sqrt{1+x^2}$ and in $\triangle AXZ$, $AZ = \sqrt{x^2-3}$. Hence $BW = 2\sqrt{x^2-3}$.

Since $AB = AC$, $YC = x + \sqrt{x^2-3}$.

By Pythagoras: in $\triangle XYC$, $XC^2 = 1 + (x+\sqrt{x^2-3})^2 = 2x^2 - 2 + 2x\sqrt{x^2-3}$
and in $\triangle XWC$, $WC^2 = 2x^2 - 18 + 2x\sqrt{x^2-3}$.

Since $BA = BC$, $2x + \sqrt{x^2-3} = 2\sqrt{x^2-3} + \sqrt{2x^2 - 18 + 2x\sqrt{x^2-3}}$.

So $2x - \sqrt{x^2-3} = \sqrt{2x^2 - 18 + 2x\sqrt{x^2-3}}$.

Squaring gives $4x^2 + x^2 - 3 - 4x\sqrt{x^2-3} = 2x^2 - 18 + 2x\sqrt{x^2-3}$, which simplifies to $3x^2 + 15 = 6x\sqrt{x^2-3}$.

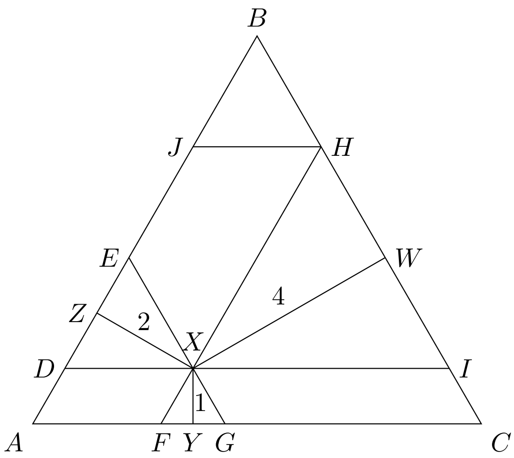
Squaring again gives $9x^4 + 90x^2 + 225 = 36x^4 - 108x^2$. So $0 = 3x^4 - 22x^2 - 25 = (3x^2 - 25)(x^2 + 1)$, giving $x = \dfrac{5}{\sqrt{3}}$.

Hence, area $AZXY = \dfrac{x}{2} + \sqrt{x^2-3} = \dfrac{5}{2\sqrt{3}} + \dfrac{4}{\sqrt{3}} = \dfrac{13}{2\sqrt{3}}$ and
area $ABC = \dfrac{\sqrt{3}}{4}(2x + \sqrt{x^2-3})^2 = \dfrac{\sqrt{3}}{4}\Big(\dfrac{10}{\sqrt{3}} + \dfrac{4}{\sqrt{3}}\Big)^2 = \dfrac{49}{\sqrt{3}}$.

Since the area of $AZXY$ is $13\,\mathrm{cm}^2$, the area of ABC is $(\dfrac{49}{\sqrt{3}}/\dfrac{13}{2\sqrt{3}}) \times 13 = \mathbf{98}\,\mathrm{cm}^2$.

Method 3

Let DI be the line through X parallel to AC with D on AB and I on BC.
Let EG be the line through X parallel to BC with E on AB and G on AC.
Let FH be the line through X parallel to AB with F on AC and H on BC.
Let J be a point on AB so that HJ is parallel to AC.
Triangles XDE, XFG, XHI, BHJ are equilateral, and triangles XDE and BHJ are congruent.



The areas of the various equilateral triangles are proportional to the square of their heights. Let the area of $\triangle FXG = 1$. Then, denoting area by $|\ |$, we have:

$|DEX| = 4$, $|XHI| = 16$, $|AEG| = 9$, $|DBI| = 36$, $|FHC| = 25$.

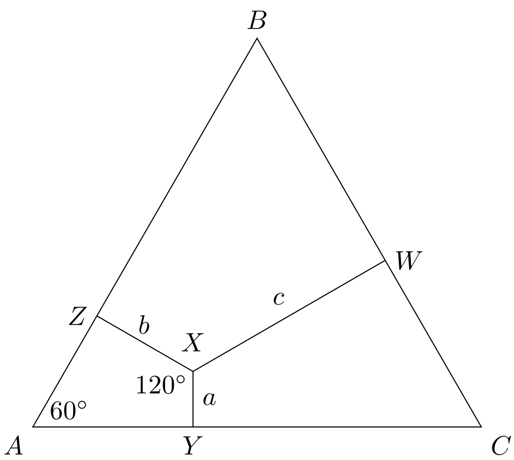
$|ABC| = |AEG| + |FHC| + |DBI| - |FXG| - |DEX| - |XHI| = 9 + 25 + 36 - 1 - 4 - 16 = 49$.

$|AZXY| = |AEG| - \frac{1}{2}(|FXG| + |DEX|) = 9 - \frac{1}{2}(1+4) = 6.5$.

Since the area of $AZXY$ is $13\,\mathrm{cm}^2$, the area of ABC is $2 \times 49 = \mathbf{98}\,\mathrm{cm}^2$.

Method 4

Consider the general case where $XY = a$, $XZ = b$, and $XW = c$.



Projecting AY onto the line through ZX gives $AY\sin 60° - a\cos 60° = b$.
Hence $AY = (a+2b)/\sqrt{3}$. Similarly, $AZ = (b+2a)/\sqrt{3}$.

Letting $|\ |$ denote area, we have

$$\begin{aligned}
|AZXY| &= |YAZ| + |YXZ| \\
&= \tfrac{1}{2}(AY)(AZ)\sin 60° + \tfrac{1}{2}ab\sin 120° \\
&= \tfrac{\sqrt{3}}{4}((AY)(AZ) + ab) \\
&= \tfrac{\sqrt{3}}{12}((a+2b)(b+2a) + 3ab) \\
&= \tfrac{\sqrt{3}}{12}(2a^2 + 2b^2 + 8ab) \\
&= \tfrac{\sqrt{3}}{6}(a^2 + b^2 + 4ab)
\end{aligned}$$

Similarly, $|CYXW| = \frac{\sqrt{3}}{6}(a^2 + c^2 + 4ac)$ and $|BWXZ| = \frac{\sqrt{3}}{6}(b^2 + c^2 + 4bc)$.

Hence $|ABC| = \frac{\sqrt{3}}{6}(2a^2 + 2b^2 + 2c^2 + 4ab + 4ac + 4bc) = \frac{\sqrt{3}}{3}(a+b+c)^2$.

So $|ABC|/|AZXY| = 2(a+b+c)^2/(a^2 + b^2 + 4ab)$.

Letting $a = k$, $b = 2k$, $c = 4k$, and $|AZXY| = 13\,\mathrm{cm}^2$, we have $|ABC| = 26(49k^2)/(k^2 + 4k^2 + 8k^2) = \mathbf{98}\,\mathrm{cm}^2$.

Investigation

Method 4 gives $|ABC|/|AZXY| = 2(a+b+c)^2/(a^2 + b^2 + 4ab)$.

Alternatively, as in Method 3,

$$\begin{aligned}
|ABC| &= |AEG| + |FHC| + |DBI| - |FXG| - |DEX| - |XHI| \\
&= (a+b)^2 + (a+c)^2 + (b+c)^2 - a^2 - b^2 - c^2 = (a+b+c)^2.
\end{aligned}$$

Also
$$\begin{aligned}
|AZXY| &= |AEG| - \tfrac{1}{2}(|FXG| + |DEX|) \\
&= (a+b)^2 - \tfrac{1}{2}(a^2 + b^2) \\
&= 2ab + \tfrac{1}{2}(a^2 + b^2).
\end{aligned}$$

So $|ABC|/|AZXY| = 2(a+b+c)^2/(a^2 + b^2 + 4ab)$.

BẢNG PHÂN TÍCH KẾT QUẢ CHI TIẾT VÀ

DANH SÁCH HỌC SINH ĐẠT CHỨNG NHẬN ``GIẢI THƯỞNG’’ VÀ ‘‘XUẤT SẮC’’

NĂM 2015

AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD STATISTICS

Distribution of Awards/School Year

Year	Number of Students	Number of Awards				
		Prize	High Distinction	Distinction	Credit	Participation
8	341	3	17	39	86	196
9	414	8	45	61	99	201
10	462	11	52	89	139	171
Other	221	4	9	16	41	151
Total	1438	26	123	205	365	719

Number of Correct Answers Questions 1–8

Year	Number Correct/Question							
	1	2	3	4	5	6	7	8
8	119	231	282	164	128	79	48	50
9	144	298	347	224	187	138	82	86
10	176	341	377	297	219	208	103	104
Other	66	132	176	81	75	34	33	21
Total	505	1002	1182	766	609	459	266	261

Mean Score/Question/School Year

Year	Number of Students	Mean Score			Overall Mean
		Question			
		1–8	9	10	
8	341	10.1	0.5	0.2	10.9
9	414	11.8	0.9	0.5	13.2
10	462	13.0	1.1	0.6	14.6
Other	221	8.6	0.4	0.2	9.3
All Years	1438	11.3	0.8	0.4	12.5

AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD RESULTS

NAME	SCHOOL	YEAR	SCORE
PRIZE			
Matthew Cheah	Penleigh and Essendon Grammar School, VIC	10	35
Puhua Cheng	Raffles Institution, Singapore	8	35
Ariel Pratama Junaidi	Anglo-Chinese School, Singapore	10	35
Evgeni Kayryakov	Childrens Academy 21st Century, Bulgaria	7	35
Wei Khor Jun	Raffles Institution, Singapore	8	35
Jack Liu	Brighton Grammar, VIC	9	35
Jerry Mao	Caulfield Grammar School, Wheelers Hill,VIC	9	35
Liao Meng	Anglo-Chinese School, Singapore	10	35
Nguyen Hoai Nam	Anglo-Chinese School, Singapore	10	35
Aloysius Ng Yangyi	Raffles Institution, Singapore	7	35
Kohsuke Sato	Christ Church Grammar School, WA	10	35
Yuelin Shen	Scotch College, WA	10	35
Chen Tan Xu	Raffles Institution, Singapore	7	35
Kit Victor Loh Wai	Raffles Institution, Singapore	8	35
Jianzhi Wang	Raffles Institution, Singapore	9	35
Zhe Xin	Raffles Institution, Singapore	9	35
Austin Zhang	Sydney Grammar School, NSW	10	35
Yu Zhiqiu	Anglo-Chinese School, Singapore	10	35
Lin Zien	Anglo-Chinese School, Singapore	9	35
Bobby Dey	James Ruse Agricultural High School, NSW	10	34
Goh Ethan	Raffles Institution, Singapore	7	34
Yulong Guo	Hwa Chong Institution, Singapore	9	34
Edwin Winata Hartanto	Anglo-Chinese School, Singapore	10	34
Hristo Papazov	Childrens Academy 21st Century, Bulgaria	10	34
Zhang Yansheng	Chung Cheng High School, Singapore	9	34
Guowen Zhang	St Joseph's College, QLD	9	34
HIGH DISTINCTION			
Ivan Ganev	Childrens Academy 21st Century, Bulgaria	10	33
Theodore Leebrant	Anglo-Chinese School, Singapore	9	33
Yu Peng Ng	Hwa Chong Institution, Singapore	8	33
Cheng Shi	Hwa Chong Institution, Singapore	8	33
Kean Tan Wee	Raffles Institution, Singapore	7	33
Sharvil Kesarwani	Merewether High School, NSW	8	32
Hong Rui Benjamin Lee	Hwa Chong Institution, Singapore	10	32
Chenxu Li	Raffles Institution, Singapore	8	32
William Li	Barker College, NSW	9	32

NAME	SCHOOL	YEAR	SCORE
Han Yang	Hwa Chong Institution, Singapore	9	32
Stanley Zhu	Melbourne Grammar School, VIC	9	32
Anand Bharadwaj	Trinity Grammar School, VIC	9	31
William Hu	Christ Church Grammar School, WA	9	31
Xianyi Huang	Baulkham Hills High School, NSW	10	31
Wanzhang Jing	James Ruse Agricultural High School, NSW	10	31
Yuhao Li	Hwa Chong Institution, Singapore	9	31
Steven Lim	Hurlstone Agricultural High School, NSW	9	31
John Min	Baulkham Hills High School, NSW	9	31
Elliott Murphy	Canberra Grammar School, ACT	10	31
Longxuan Sun	Hwa Chong Institution, Singapore	8	31
Boyan Wang	Hwa Chong Institution, Singapore	9	31
Sean Zammit	Barker College, NSW	10	31
Atul Barman	James Ruse Agricultural High School, NSW	10	30
Atanas Dinev	Childrens Academy 21st Century, Bulgaria	9	30
Bill Hu	James Ruse Agricultural High School, NSW	10	30
Phillip Huynh	Brisbane State High School, QLD	10	30
Wei Ci William Kin	Hwa Chong Institution, Singapore	10	30
Yang Lee Ker	Raffles Institution, Singapore	9	30
Forbes Mailler	Canberra Grammar School, ACT	9	30
Moses Mayer	Surya Institute, Indonesia	9	30
Zlatina Mileva	Childrens Academy 21st Century, Bulgaria	8	30
Kirill Saulov	Brisbane Grammar School, QLD	10	30
Yuxuan Seah	Raffles Institution, Singapore	8	30
Kieran Shivakumaarun	Sydney Boys High School, NSW	10	30
Liang Tan Xue	Raffles Institution, Singapore	10	30
Nicholas Tanvis	Anglo-Chinese School, Singapore	9	30
An Aloysius Wang	Hwa Chong Institution, Singapore	10	30
William Wang	Queensland Academy for Science, Mathematics and Technology, QLD	10	30
Joshua Welling	Melrose High School, ACT	9	30
Seung Hoon Woo	Hwa Chong Institution, Singapore	10	30
Chen Yanbing	Methodist Girls' School, Singapore	10	30
Guangxuan Zhang	Raffles Institution, Singapore	10	30
Chi Zhang Yu	Raffles Institution, Singapore	7	30
Keer Chen	Presbyterian Ladies' College, NSW	10	29
Linus Cooper	James Ruse Agricultural High School, NSW	9	29
Liam Coy	Sydney Grammar School, NSW	7	29
Rong Dai Xiang	Raffles Institution, Singapore	8	29
Hong Pei Goh	Hwa Chong Institution, Singapore	10	29

NAME	SCHOOL	YEAR	SCORE
Tianjie Huang	Hwa Chong Institution, Singapore	9	29
Ricky Huang	James Ruse Agricultural High School, NSW	10	29
Tianjie Huang	Hwa Chong Institution, Singapore	9	29
Yu Jiahuan	Raffles Girls' School, Singapore	9	29
Tony Jiang	Scotch College, VIC	10	29
Charles Li	Camberwell Grammar School, Vic	9	29
Steven Liu	James Ruse Agricultural High School, NSW	10	29
Hilton Nguyen	Sydney Technical High School, NSW	9	29
James Nguyen	Baulkham Hills High School, NSW	10	29
Trung Nguyen	Penleigh and Essendon Grammar School, VIC	10	29
James Phillips	Canberra Grammar School, ACT	9	29
Ryan Stocks	Radford College, ACT	9	29
Hadyn Tang	Trinity Grammar School, VIC	6	29
Stanve Avrilium Widjaja	Surya Institute, Indonesia	7	29
Wang Yihe	Anglo-Chinese School, Singapore	9	29
Sun Yue	Raffles Girls' School, Singapore	9	29
Wang Beini	Raffles Girls' School, Singapore	10	28
Chwa Channe	Raffles Girls' School, Singapore	8	28
Keiran Hamley	All Saints Anglican School, QLD	10	28
Lee Shi Hao	Anglo-Chinese School, Singapore	9	28
Zhu Jiexiu	Raffles Girls' School, Singapore	9	28
Jodie Lee	Seymour College, SA	10	28
Yu Hsin Lee	Hwa Chong Institution, Singapore	9	28
Phillip Liang	James Ruse Agricultural High School, NSW	9	28
Anthony Ma	Shore School, NSW	10	28
Dzaki Muhammad	Surya Institute, Indonesia	9	28
Daniel Qin	Scotch College, VIC	10	28
Sang Ta	Randwick Boys High School, NSW	8	28
Ruiqian Tong	Presbyterian Ladies’ College, VIC	9	28
Hu Xing Yi	Methodist Girls' School, Singapore	10	28
Zhao Yiyang	Methodist Girls' School, Singapore	10	28
Claire Yung	Lyneham High School, ACT	10	28
Xuan Ang Ben	Raffles Institution, Singapore	9	27
Hantian Chen	James Ruse Agricultural High School, NSW	10	27
Jasmine Jiawei Chen	Pymble Ladies’ College, NSW	10	27
Harry Dinh	James Ruse Agricultural High School, NSW	10	27
Tan Jian Yee	Chung Cheng High School, Singapore	10	27
Daniel Jones	All Saints Anglican Senior School, QLD	10	27
Winfred Kong	Hwa Chong Institution, Singapore	10	27
Adrian Law	James Ruse Agricultural High School, NSW	10	27

NAME	SCHOOL	YEAR	SCORE
Jason Leung	James Ruse Agricultural High School, NSW	7	27
Sabrina Natashya Liandra	Surya Institute, Indonesia	9	27
Angela (Yunyun) Ran	The Mac.Robertson Girls' High School, VIC	9	27
Yi Shen Xin	Raffles Institution, Singapore	7	27
Peter Tong	Yarra Valley Grammar, VIC	9	27
Jordan Truong	Sydney Technical High School, NSW	10	27
Andrew Virgona	Concord High School, NSW	9	27
Tommy Wei	Scotch College, VIC	9	27
Tianyi Xu	Sydney Boys High School, NSW	9	27
Wu Zhen	Anglo-Chinese School, Singapore	10	27
Gordon Zhuang	Sydney Boys High School, NSW	9	27
Zhou Zihan	Raffles Girls' School, Singapore	8	27
Amit Ben-Harim	McKinnon Secondary College, VIC	9	26
Hu Chen	The King's School, NSW	10	26
Merry Xiao Die Chu	North Sydney Girls' High School, NSW	9	26
Li Haocheng	Anglo-Chinese School, Singapore	9	26
William Hu	Rossmoyne Senior High School, WA	10	26
Laeeque Jamdar	Baulkham Hills High School, NSW	9	26
Yasiru Jayasoora	James Ruse Agricultural High School, NSW	7	26
Arun Jha	Perth Modern School, WA	10	26
Tony Li	Sydney Boys High School, NSW	10	26
Zefeng Jeff Li	Glen Waverley Secondary College, VIC	8	26
Adrian Lo	Newington College, NSW	7	26
Lionel Maizels	Norwood Secondary College, VIC	8	26
Marcus Rees	Taroona High School, TAS	8	26
Elva Ren	Presbyterian Ladies College, VIC	10	26
Aidan Smith	All Saints' College, WA	8	26
Jacob Smith	All Saints' College, WA	9	26
Keane Teo	Hwa Chong Institution, Singapore	10	26
Jeffrey Wang	Shore School, NSW	10	26
Xinlu Xu	Presbyterian Ladies' College, Sydney, NSW	10	26
Jason (Yi) Yang	James Ruse Agricultural High School, NSW	8	26
Shukai Zhang	Hwa Chong Institution, Singapore	10	26
Yanjun Zhang	Hwa Chong Institution, Singapore	9	26
Kevin Zhu	James Ruse Agricultural High School, NSW	10	26
Jonathan Zuk	Elwood College, VIC	8	26

ĐỀ THI VÀ ĐÁP ÁN CHI TIẾT

CUỘC THI VÔ ĐỊCH TOÁN CẤP TRUNG HỌC ÚC MỞ RỘNG NĂM 2014

Australian Mathematical Olympiad Committee

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

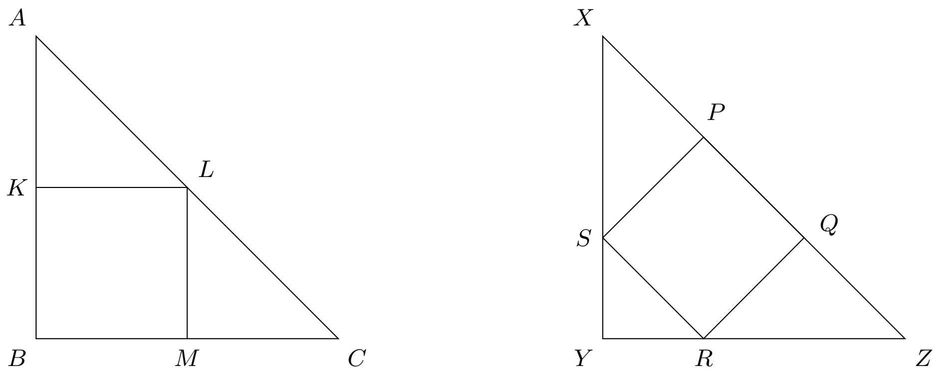
Australian Intermediate Mathematics Olympiad 2014

Questions

1. In base b, the square of 24_b is 521_b. Find the value of b in base 10.

[**2 marks**]

2. Triangles ABC and XYZ are congruent right-angled isosceles triangles. Squares $KLMB$ and $PQRS$ are as shown. If the area of $KLMB$ is 189, find the area of $PQRS$.



[**2 marks**]

3. Let x and y be positive integers that simultaneously satisfy the equations $xy = 2048$ and $\frac{x}{y} - \frac{y}{x} = 7.875$. Find x.

[**3 marks**]

4. Joel has a number of blocks, all with integer weight in kilograms. All the blocks of one colour have the same weight and blocks of a different colour have different weights.

Joel finds that various collections of some of these blocks have the same total weight w kg. These collections include:

1. 5 red, 3 blue and 5 green;
2. 4 red, 5 blue and 4 green;
3. 7 red, 4 blue and some green.

If $30 < w < 50$, what is the total weight in kilograms of 6 red, 7 blue and 3 green blocks?

[**3 marks**]

5. Let $\frac{1}{a} + \frac{1}{b} = \frac{1}{20}$, where a and b are positive integers. Find the largest value of $a + b$.

[**4 marks**]

PLEASE TURN OVER THE PAGE FOR QUESTIONS 6, 7, 8, 9 AND 10

6. Justin's sock drawer contains only identical black socks and identical white socks, a total of less than 50 socks altogether.

If he withdraws two socks at random, the probability that he gets a pair of the same colour is 0.5. What is the largest number of black socks he can have in his drawer?

[**4 marks**]

7. A *code* is a sequence of 0s and 1s that does not have three consecutive 0s. Determine the number of codes that have exactly 11 digits.

[**4 marks**]

8. Determine the largest integer n which has at most three digits and equals the remainder when n^2 is divided by 1000.

[**4 marks**]

9. Let $ABCD$ be a trapezium with $AB \parallel CD$ such that

(i) its vertices A, B, C, D, lie on a circle with centre O,

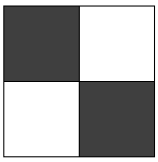
(ii) its diagonals AC and BD intersect at point M and $\angle AMD = 60°$,

(iii) $MO = 10$.

Find the difference between the lengths of AB and CD.

[**5 marks**]

10. An $n \times n$ grid with $n > 1$ is covered by several copies of a 2×2 square tile as in the figure below. Each tile covers precisely four cells of the grid and each cell of the grid is covered by at least one cell of one tile. The tiles may be rotated 90 degrees.



(a) Show there exists a covering of the grid such that there are exactly n black cells visible.
(b) Prove there is no covering where there are less than n black cells visible.
(c) Determine the maximum number of visible black cells.

[**4 marks**]

Investigation

(i) Show that, for each possible pattern of 3 black cells and 6 white cells on a 3×3 grid, there is a covering whose visible cells have that pattern. [**1 bonus mark**]

(ii) Explain why not all patterns of 4 black cells and 12 white cells on a 4×4 grid can be achieved with a covering in which each new tile must be placed on top of all previous tiles that it overlaps. [**1 bonus mark**]

(iii) Determine the maximum number of visible black cells for a covering of an $n \times m$ grid where $1 < n < m$. [**2 bonus marks**]

The Mathematics Olympiads are supported by the Australian Government
Department of Education through the Mathematics and Science Participation Program.

AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

Australian Intermediate Mathematics Olympiad 2014

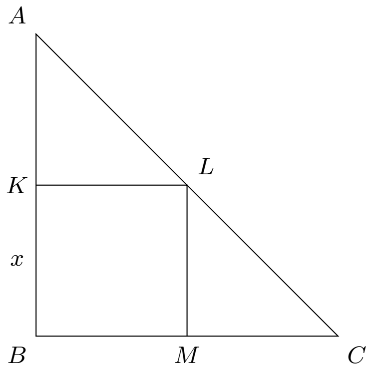
Solutions

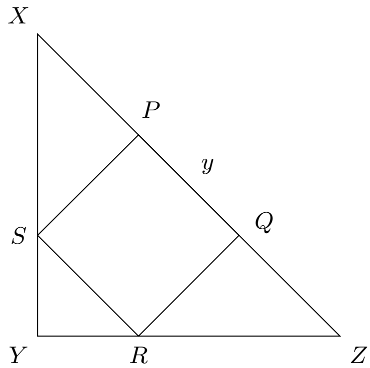
1. We have $24_b = 2b + 4$, $521_b = 5b^2 + 2b + 1$ and $521_b = (2b + 4)^2 = 4b^2 + 16b + 16$. $\boxed{1}$

Hence $0 = b^2 - 14b - 15 = (b - 15)(b + 1)$. Therefore $b = \mathbf{15}$. $\boxed{1}$

2. *Preamble for Methods 1, 2, 3*

Let $BK = x$ and $PQ = y$.





Since ABC is a right-angled isosceles triangle and $BMLK$ is a square, CML and AKL are also right-angled isosceles triangles. Therefore $AK = CM = x$.

Since XYZ is a right-angled isosceles triangle and $PQRS$ is a square, XPS and ZQR and therefore YRS are also right-angled isosceles triangles. Therefore $XP = ZQ = y$. $\boxed{1}$

Method 1

We have $3y = XZ = AC = AB\sqrt{2} = 2x\sqrt{2}$. So $y = \frac{2\sqrt{2}}{3}x$.
Hence the area of $PQRS = y^2 = \frac{8}{9}x^2 = \frac{8}{9} \times 189 = \mathbf{168}$. $\boxed{1}$

Method 2

We have $2x = AB = \frac{AC}{\sqrt{2}} = \frac{XZ}{\sqrt{2}} = \frac{3y}{\sqrt{2}}$. So $y = \frac{2\sqrt{2}}{3}x$.
Hence the area of $PQRS = y^2 = \frac{8}{9}x^2 = \frac{8}{9} \times 189 = \mathbf{168}$. $\boxed{1}$

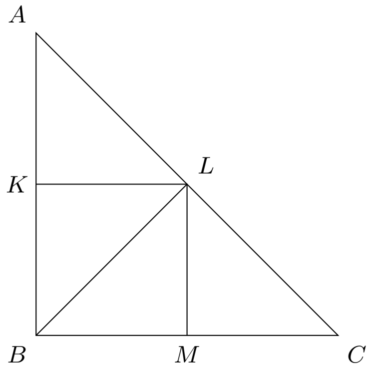
Method 3

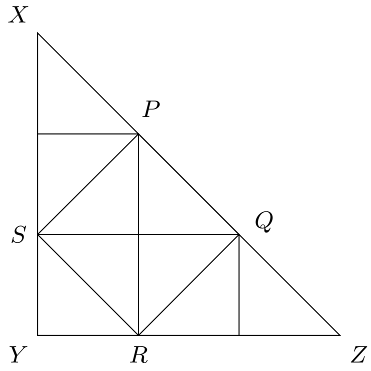
We have $2x = AB = XY = XS + SY = \sqrt{2}y + \frac{1}{\sqrt{2}}y = (\sqrt{2} + \frac{1}{\sqrt{2}})y = \frac{3}{\sqrt{2}}y$. So $y = \frac{2\sqrt{2}}{3}x$.
Hence the area of $PQRS = y^2 = \frac{8}{9}x^2 = \frac{8}{9} \times 189 = \mathbf{168}$. $\boxed{1}$

Method 4

Joining B to L divides $\triangle ABC$ into 4 congruent right-angled isosceles triangles. Hence the area of $\triangle ABC$ is twice the area of $KLMB$.

Drawing the diagonals of $PQRS$ and the perpendiculars from P to XS and from Q to RZ divides $\triangle XYZ$ into 9 congruent right-angled isosceles triangles.





Hence the area of $PQRS = \frac{4}{9}\times$area of $\triangle XYZ = \frac{4}{9}\times$area of $\triangle ABC = \frac{4}{9}\times 2\times$area of $KLMB = \frac{8}{9}\times 189 = \mathbf{168}$. $\boxed{1}$

3. *Preamble for Methods 1, 2, 3*

Since x, y, and $\frac{x}{y}-\frac{y}{x}$ are all positive, we know that $x > y$. Since $xy = 2048 = 2^{11}$ and x and y are integers, we know that x and y are both powers of 2. $\boxed{1}$

Method 1

Therefore $(x,y) = (2048,1), (1024,2), (512,4), (256,8), (128,16)$, or $(64,32)$. $\boxed{1}$

Only (128,16) satisfies $\frac{x}{y}-\frac{y}{x} = 7.785 = 7\frac{7}{8} = \frac{63}{8}$. So $x = \mathbf{128}$. $\boxed{1}$

Method 2

Let $x = 2^m$ and $y = 2^n$. Then $m > n$ and $xy = 2^{m+n}$, so $m + n = 11$.
From $\frac{x}{y}-\frac{y}{x} = 7.785 = 7\frac{7}{8}$ we have $2^{m-n} - 2^{n-m} = \frac{63}{8}$.
Let $m - n = t$. Then $2^t - 2^{-t} = \frac{63}{8}$. So $0 = 8(2^t)^2 - 63(2^t) - 8 = (2^t - 8)(8(2^t) + 1)$. $\boxed{1}$

Hence $2^t = 8 = 2^3$, $m - n = 3$, $2m = 14$, and $m = 7$. Therefore $x = 2^7 = \mathbf{128}$. $\boxed{1}$

Method 3

Let $x = 2^m$ and $y = 2^n$. Then $m > n$.
From $\frac{x}{y}-\frac{y}{x} = 7.875 = 7\frac{7}{8}$ we have $x^2 - y^2 = \frac{63}{8}xy = \frac{63}{8}\times 2048 = 63\times 2^8$.
So $63\times 2^8 = (x-y)(x+y) = (2^m - 2^n)(2^m + 2^n) = 2^{2n}(2^{m-n} - 1)(2^{m-n} + 1)$. $\boxed{1}$

Hence $2^{2n} = 2^8$, $2^{m-n} - 1 = 7$, and $2^{m-n} + 1 = 9$.
Therefore $n = 4$, $2^{m-4} = 8$, and $x = 2^m = 8\times 2^4 = 2^7 = \mathbf{128}$. $\boxed{1}$

Method 4

Now $\frac{x}{y} - \frac{y}{x} = 7.875 = 7\frac{7}{8} = \frac{63}{8}$ and $\frac{x}{y} - \frac{y}{x} = \frac{x^2-y^2}{xy} = \frac{x^2-y^2}{2048}$.
So $x^2 - y^2 = \frac{63}{8} \times 2048 = 63 \times 2^8 = (64-1)2^8 = (2^6-1)2^8 = 2^{14} - 2^8$. $\boxed{1}$

Substituting $y = 2048/x = 2^{11}/x$ gives $x^2 - 2^{22}/x^2 = 2^{14} - 2^8$.
Hence $0 = (x^2)^2 - (2^{14} - 2^8)x^2 - 2^{22} = (x^2 - 2^{14})(x^2 + 2^8)$. $\boxed{1}$

So $x^2 = 2^{14}$. Since x is positive, $x = 2^7 = \mathbf{128}$. $\boxed{1}$

Method 5

We have $\frac{x}{y} - \frac{y}{x} = 7.875 = 7\frac{7}{8} = \frac{63}{8}$. Multiplying by xy gives $x^2 - y^2 = \frac{63}{8}xy$. $\boxed{1}$

So $8x^2 - 63xy - 8y^2 = 0$ and $(8x + y)(x - 8y) = 0$. $\boxed{1}$

Since x and y are positive, $x = 8y$, $8y^2 = xy = 2048$, $y^2 = 256$, $y = 16$, $x = \mathbf{128}$. $\boxed{1}$

Comment. From Method 4 or 5, we don't need to know that x and y are integers to solve this problem.

4. Let the red, blue and green blocks have different weights r, b and g kg respectively.
Then we have:

$$\begin{aligned} 5r + 3b + 5g &= w \qquad (1)\\ 4r + 5b + 4g &= w \qquad (2)\\ 7r + 4b + ng &= w \qquad (3) \end{aligned}$$

where n is the number of green blocks. $\boxed{1}$

Subtracting (1) and (2) gives $2b = r + g$.
Substituting in (2) gives $13b = w$, so w is a multiple of 13 between 30 and 50.
Hence $w = 39$, $b = 3$ and $r + g = 6$. $\boxed{1}$

Method 1

Since $r + g = 6$, r is one of the numbers 1, 2, 4, 5.
If r is 4 or 5, $7r + 4b > 39$ and (3) cannot be satisfied.
If $r = 2$, then $g = 4$ and (3) gives $26 + 4n = 39$, which cannot be satisfied in integers.
So $r = 1$, then $g = 5$ and (3) gives $19 + 5n = 39$ and $n = 4$.
Hence the total weight in kilograms of 6 red, 7 blue, and 3 green blocks is
$6 \times 1 + 7 \times 3 + 3 \times 5 = \mathbf{42}$. $\boxed{1}$

Method 2

Since $r + g = 6$, g is one of the numbers 1, 2, 4, 5.
Substituting $r = 6 - g$ in (3) gives $(7 - n)g = 15$. Thus g is 1 or 5.
If $g = 1$, then $n = -8$, which is not allowed.
If $g = 5$, then $n = 4$ and $r = 1$.
Hence the total weight in kilograms of 6 red, 7 blue, and 3 green blocks is
$6 \times 1 + 7 \times 3 + 3 \times 5 = \mathbf{42}$. $\boxed{1}$

5. *Method 1*

From symmetry we may assume $a \le b$. If $a = b$, then both are 40 and $a + b = 80$. We now assume $a < b$. As a increases, b must decrease to satisfy the equation $\frac{1}{a} + \frac{1}{b} = \frac{1}{20}$. So $a < 40$. $\boxed{1}$

We have $\frac{1}{b} = \frac{1}{20} - \frac{1}{a} = \frac{a-20}{20a}$. So $b = \frac{20a}{a-20}$. Since a and b are positive, $a > 20$. $\boxed{1}$

The table shows all integer values of a and b.

a	21	22	24	25	28	30	36
b	420	220	120	100	70	60	45

$\boxed{1}$

Thus the largest value of $a + b$ is $21 + 420 = \mathbf{441}$. $\boxed{1}$

Method 2

As in Method 1, we have $b = \frac{20a}{a-20}$ and $20 < a \le 40$. $\boxed{2}$

So $a + b = a(1 + \frac{20}{a-20})$.

If $a = 21$, than $a + b = 21(1 + 20) = 441$. If $a \ge 22$, then $a + b \le 40(1 + 10) = 440$. $\boxed{1}$

Thus the largest value of $a + b$ is **441**. $\boxed{1}$

Method 3

We have $ab = 20(a + b)$. So $(a - 20)(b - 20) = 400 = 2^4 5^2$. $\boxed{1}$

Since b is positive, $ab > 20a$ and $a > 20$. Similarly $b > 20$. $\boxed{1}$

From symmetry we may assume $a \le b$ hence $a - 20 \le b - 20$. $\boxed{1}$

The table shows all values of $a - 20$ and the corresponding values of $b - 20$.

$a - 20$	1	2	4	8	16	5	10	20
$b - 20$	400	200	100	50	25	80	40	20

Thus the largest value of $a + b$ is $21 + 420 = \mathbf{441}$. $\boxed{1}$

Method 4

We have $ab = 20(a + b)$, so 5 divides a or b. Since b is positive, $ab > 20a$ and $a > 20$.

Suppose 5 divides a and b. From symmetry we may assume $a \le b$. The following table gives all values of a and b.

a	25	30	40
b	100	60	40

$\boxed{1}$

Suppose 5 divides a but not b. Since $b(a - 20) = 20a$, 25 divides $a - 20$. Let $a = 20 + 25n$. Then $(20 + 25n)b = 20(20 + 25n + b)$, $nb = 16 + 20n$, $n(b - 20) = 16$. The following table gives all values of n, b, and a.

n	1	2	4	8	16
$b - 20$	16	8	4	2	1
b	36	28	24	22	21
a	45	70	120	220	420

$\boxed{1}$

A similar table is obtained if 5 divides b but not a. $\boxed{1}$

Thus the largest value of $a + b$ is $21 + 420 = \mathbf{441}$. $\boxed{1}$

Method 5

We have $ab = 20(a+b)$. So maximising $a+b$ is equivalent to maximising ab, which is equivalent to minimising $\frac{1}{ab}$. $\boxed{1}$

Let $x = \frac{1}{a}$ and $y = \frac{1}{b}$. We want to minimise xy subject to $x + y = \frac{1}{20}$. From symmetry we may assume $x \geq y$. Hence $x \geq \frac{1}{40}$. $\boxed{1}$

Thus we want to minimise $z = x(\frac{1}{20} - x)$ with $z > 0$, hence with $0 < x < \frac{1}{20}$. The graph of this function is an inverted parabola with its turning point at $x = \frac{1}{40}$. So the minimum occurs at $x = \frac{1}{21}$. This corresponds to $y = \frac{1}{20} - \frac{1}{21} = \frac{1}{420}$. $\boxed{1}$

Thus the largest value of $a + b$ is $21 + 420 = \mathbf{441}$. $\boxed{1}$

6. *Method 1*

Let b be the number of black socks and w the number of white ones. If b or w is 0, then the probability of withdrawing a pair of socks of the same colour would be 1. So b and w are positive. From symmetry we may assume that $b \geq w$.

The number of pairs of black socks is $b(b-1)/2$. The number of pairs of white socks is $w(w-1)/2$. The number of pairs of socks with one black and the other white is bw. $\boxed{1}$

The probability of selecting a pair of socks of the same colour is the same as the probability of selecting a pair of socks of different colour. Hence $b(b-1)/2 + w(w-1)/2 = bw$ or

$$b(b-1) + w(w-1) = 2bw \qquad \boxed{1}$$

Let $d = b - w$. Then $w = b - d$ and

$$\begin{aligned} b(b-1) + (b-d)(b-d-1) &= 2b(b-d) \\ b^2 - b + b^2 - bd - b - bd + d^2 + d &= 2b^2 - 2bd \\ -2b + d^2 + d &= 0 \\ d(d+1) &= 2b \qquad \boxed{1} \end{aligned}$$

The following table shows all possible values of d. Note that $b + w = 2b - d = d^2$.

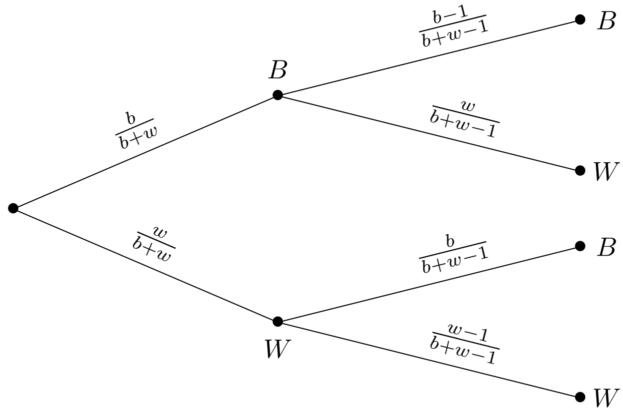
d	0	1	2	3	4	5	6	7	≥ 8
b	0	1	3	6	10	15	21	28	
$b+w$	0	1	4	9	16	25	36	49	≥ 64

Thus the largest value of b is **28**. $\boxed{1}$

Preamble for Methods 2, 3, 4

Let b be the number of black socks and w the number of white ones. If b or w is 0, then the probability of withdrawing a pair of socks of the same colour would be 1. So b and w are positive. From symmetry we may assume that $b \geq w$.

The pair of socks that Justin withdraws are either the same colour or different colours. So the probability that he draws a pair of socks of different colours is $1 - 0.5 = 0.5$. The following diagram shows the probabilities of withdrawing one sock at a time.



$\boxed{1}$

So the probability that Justin draws a pair of socks of different colours is $\frac{2bw}{(b+w)(b+w-1)}$.
Hence $4bw = b^2 + 2bw + w^2 - b - w$ and $b^2 - 2bw + w^2 - b - w = 0$. $\boxed{1}$

Method 2

We have $b^2 - (2w + 1)b + (w^2 - w) = 0$.
The quadratic formula gives $b = (2w+1 \pm \sqrt{(2w+1)^2 - 4(w^2 - w)})/2 = (2w+1 \pm \sqrt{8w+1})/2$.
If $b = (2w + 1 - \sqrt{8w+1})/2 = w + \frac{1}{2} - \frac{1}{2}\sqrt{8w+1}$, then $b \leq w + \frac{1}{2} - \frac{1}{2}\sqrt{9} = w - 1 < w$.
So $b = (2w + 1 + \sqrt{8w+1})/2$. $\boxed{1}$

Now $w < 25$ otherwise $b + w \geq 2w \geq 50$.
Since b increases with w, we want the largest value of w for which $8w + 1$ is square. Thus $w = 21$ and the largest value of b is $(42 + 1 + \sqrt{169})/2 = (43 + 13)/2 = \mathbf{28}$. $\boxed{1}$

Method 3

We have $b + w = (b - w)^2$. Thus $b + w$ is a square number less than 50 and greater than 1. $\boxed{1}$

The following tables gives all values of $b + w$ and the corresponding values of $b - w$ and b.

$b+w$	4	9	16	25	36	49
$b-w$	2	3	4	5	6	7
$2b$	6	12	20	30	42	56
b	3	6	10	15	21	28

Thus the largest value of b is **28**. $\boxed{1}$

Method 4

We have $b + w = (b - w)^2$. Also $w < 25$ otherwise $b + w \geq 2w \geq 50$. For a fixed value of w, consider the line $y = w + b$ and parabola $y = (b - w)^2$. These intersect at a unique point for $b \geq w$. For each value of w we guess and check a value of b for which the line and parabola intersect. $\boxed{1}$

w	b	$b+w$	$(b-w)^2$	$b+w = (b-w)^2$?
24	31	55	49	$b + w > (b - w)^2$
	32	56	64	$b + w < (b - w)^2$
23	30	53	49	$b + w > (b - w)^2$
	31	54	64	$b + w < (b - w)^2$
22	29	51	49	$b + w > (b - w)^2$
	30	52	64	$b + w < (b - w)^2$
21	27	48	36	$b + w > (b - w)^2$
	28	49	49	$b + w = (b - w)^2$

As w decreases, the line $y = w + b$ shifts down and the parabola $y = (b - w)^2$ shifts left so their point of intersection shifts left. So b decreases as w decreases. Thus the largest value of b is **28**. $\boxed{1}$

Comment. We have $b + w = (b - w)^2$. Let $b - w = n$. Then $b + w = n^2$. Hence $b = (n^2 + n)/2 = n(n+1)/2$ and $w = (n^2 - n)/2 = (n-1)n/2$. Thus w and b are consecutive triangular numbers.

7. *Method 1*

Let c_n be the number of codes that have exactly n digits.

For $n \geq 4$, a code with n digits ends with 1 or 10 or 100. $\boxed{1}$

If the code ends in 1, then the string that remains when the end digit is removed is also a code. So the number of codes that end in 1 and have exactly n digits equals c_{n-1}.

If the code ends in 10, then the string that remains when the last 2 digits are removed is also a code. So the number of codes that end in 10 and have exactly n digits equals c_{n-2}. $\boxed{1}$

If the code ends in 100, then the string that remains when the last 3 digits are removed is also a code. So the number of codes that end in 100 and have exactly n digits equals c_{n-3}.

Hence, for $n \geq 4$, $c_n = c_{n-1} + c_{n-2} + c_{n-3}$. $\boxed{1}$

By direct counting, $c_1 = 2$, $c_2 = 4$, $c_3 = 7$. The table shows c_n for $1 \leq n \leq 11$.

n	1	2	3	4	5	6	7	8	9	10	11
c_n	2	4	7	13	24	44	81	149	274	504	927

Thus the number of codes that have exactly 11 digits is **927**. $\boxed{1}$

Method 2

Let c_n be the number of codes that have exactly n digits.

A code ends with 0 or 1. $\boxed{1}$

Suppose $n \geq 5$. If a code ends with 1, then the string that remains when the end digit is removed is also a code. So the number of codes that end with 1 and have exactly n digits equals c_{n-1}. $\boxed{1}$

If a code with n digits ends in 0, then the string that remains when the end digit is removed is a code with $n-1$ digits that does not end with two 0s. If a code with $n-1$ digits ends with two 0s, then it ends with 100. If the 100 is removed then the string that remains is an unrestricted code that has exactly $n-4$ digits. So the number of codes with $n-1$ digits that do not end with two 0s is $c_{n-1} - c_{n-4}$.

Hence, for $n \geq 5$, $c_n = 2c_{n-1} - c_{n-4}$. $\boxed{1}$

By direct counting, $c_1 = 2$, $c_2 = 4$, $c_3 = 7$, $c_4 = 13$. The table shows c_n for $1 \leq n \leq 11$.

n	1	2	3	4	5	6	7	8	9	10	11
c_n	2	4	7	13	24	44	81	149	274	504	927

Thus the number of codes that have exactly 11 digits is **927**. $\boxed{1}$

Comment. The equation $c_n = 2c_{n-1} - c_{n-4}$ can also be derived from the equation
$c_n = c_{n-1} + c_{n-2} + c_{n-3}$.
For $n \geq 5$ we have $c_{n-1} = c_{n-2} + c_{n-3} + c_{n-4}$.
Hence $c_n = c_{n-1} + (c_{n-1} - c_{n-4}) = 2c_{n-1} - c_{n-4}$.

8. *Method 1*

The square of n has the same last three digits of n if and only if $n^2 - n = n(n-1)$ is divisible by $1000 = 2^3 \times 5^3$. $\boxed{1}$

As n and $n-1$ are relatively prime, only one of those two numbers is even and only one of them can be divisible by 5. This yields the following cases.

Case 1. n is divisible by both 2^3 and 5^3. Then $n \geq 1000$, a contradiction.

Case 2. $n-1$ is divisible by both 2^3 and 5^3. Then $n \geq 1001$, a contradiction. $\boxed{1}$

Case 3. n is divisible by 2^3 and $n-1$ is divisible by 5^3. The second condition implies that n is one of the numbers 1, 126, 251, 376, 501, 626, 751, 876. Since n is also divisible by 8, this leaves $n = 376$. $\boxed{1}$

Case 4. n is divisible by 5^3 and $n-1$ is divisible by 2^3. The first condition implies that n is one of the numbers 125, 250, 375, 500, 625, 750, 875. But n must also leave remainder 1 when divided by 8, which leaves $n = 625$.

Therefore $n = \mathbf{625}$. $\boxed{1}$

Method 2

We want a number n and its square to have the same last three digits.

First, n and n^2 should have the same last digit. Squaring each of the digits from 0 to 9 shows that the last digit of n must be 0, 1, 5 or 6. $\boxed{1}$

Second, n and n^2 should have the same last two digits. Squaring each of the 2-digit numbers 00 to 90, 01 to 91, 05 to 95, and 06 to 96 as in the following table shows that the last two digits of n must be 00, 01, 25 or 76.

n	n^2	n	n^2	n	n^2	n	n^2
00	00	01	01	05	25	06	36
10	00	11	21	15	25	16	56
20	00	21	41	25	25	26	76
30	00	31	61	35	25	36	96
40	00	41	81	45	25	46	16
50	00	51	01	55	25	56	36
60	00	61	21	65	25	66	56
70	00	71	41	75	25	76	76
80	00	81	61	85	25	86	96
90	00	91	81	95	25	96	16

$\boxed{1}$

Finally, n and n^2 should have the same last three digits. Squaring each of the 3-digit numbers 000 to 900, 001 to 901, 025 to 925, and 076 to 976 as in the following table shows that the last three digits of n must be 000, 001, 625 or 376.

n	n^2	n	n^2	n	n^2	n	n^2
000	000	001	001	025	625	076	776
100	000	101	201	125	625	176	976
200	000	201	401	225	625	276	176
300	000	301	601	325	625	376	376
400	000	401	801	425	625	476	576
500	000	501	001	525	625	576	776
600	000	601	201	625	625	676	976
700	000	701	401	725	625	776	176
800	000	801	601	825	625	876	376
900	000	901	801	925	625	976	576

$\boxed{1}$

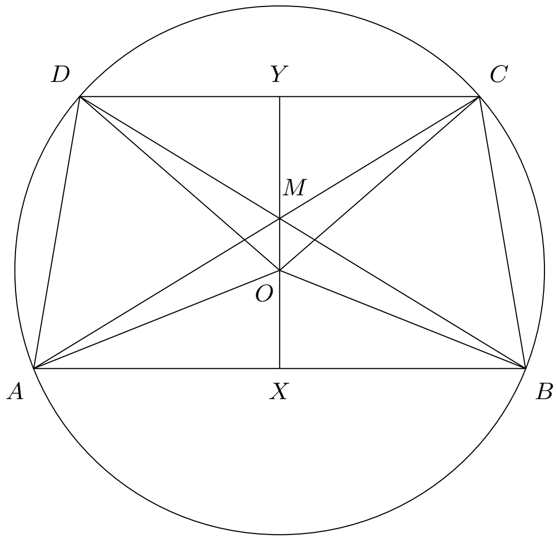
Therefore $n =$ **625**. $\boxed{1}$

9. *Preamble*

Since $ABCD$ is a cyclic quadrilateral, $\angle DCA = \angle DBA$. Since $AB \parallel CD$, $\angle DCA = \angle CAB$. So $\triangle AMB$ is isosceles. Similarly $\triangle CMD$ is isosceles. $\boxed{1}$

Extend MO to intersect AB at X and CD at Y.

Since $OA = OB$, triangles AMO and BMO are congruent. So $\angle AMO = \angle BMO$. Since $\angle AMD = 60°$, $\angle AMB = 120°$ and $\angle AMO = \angle BMO = 60°$. Hence triangles AMX and BMX are congruent and have angles 30°, 60°, 90°. Similarly DMY and CMY are congruent 30-60-90 triangles. $\boxed{1}$



Method 1

We know that X and Y are the midpoints of AB and CD respectively. Let $2x$ and $2y$ be the lengths of AB and CD respectively. From the 30-90-60 triangles AXM and CYM we have $XM = \frac{x}{\sqrt{3}}$ and $YM = \frac{y}{\sqrt{3}}$. $\boxed{1}$

From the right-angled triangles AXO and CYO, Pythagoras gives

$$AO^2 = x^2 + (\frac{x}{\sqrt{3}} - 10)^2 = \frac{4}{3}x^2 + 100 - \frac{20}{\sqrt{3}}x$$

$$CO^2 = y^2 + (\frac{y}{\sqrt{3}} + 10)^2 = \frac{4}{3}y^2 + 100 + \frac{20}{\sqrt{3}}y$$

$\boxed{1}$

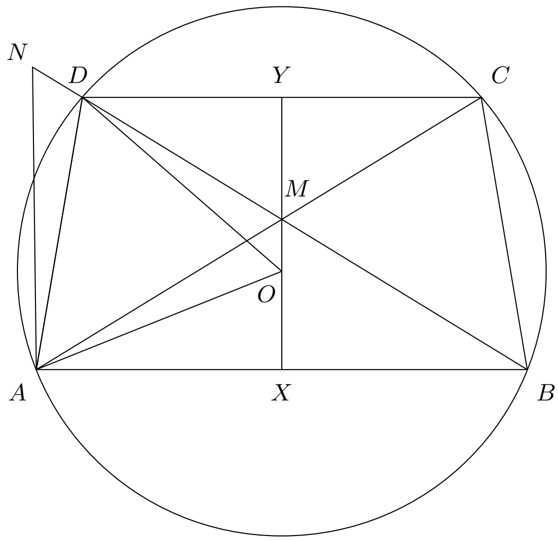
These equations also hold if O lies outside the trapezium $ABCD$.

Since $AO = CO$, we have $\frac{4}{3}(x^2 - y^2) = \frac{20}{\sqrt{3}}(x + y)$, $x^2 - y^2 = 5\sqrt{3}(x + y)$, $x - y = 5\sqrt{3}$ and $AB - CD = 2(x - y) = 10\sqrt{3}$. $\boxed{1}$

Method 2

We know that $\angle ABD = 30°$. Since O is the centre of the circle we have $\angle AOD = 2\angle ABD = 60°$. Thus $\angle AOD = \angle AMD$, hence $AOMD$ is cyclic. Since $OA = OD$ and $\angle AOD = 60°$, $\triangle AOD$ is equilateral. $\boxed{1}$

Rotate $\triangle AOM$ $60°$ anticlockwise about A to form triangle ADN.



Since $AOMD$ is cyclic, $\angle AOM + \angle ADM = 180°$. Hence MDN is a straight line. Since $\angle AMD = 60°$ and $AM = AN$, $\triangle AMN$ is equilateral. So $AM = MN = MD + DN = MD + MO$. $\boxed{1}$

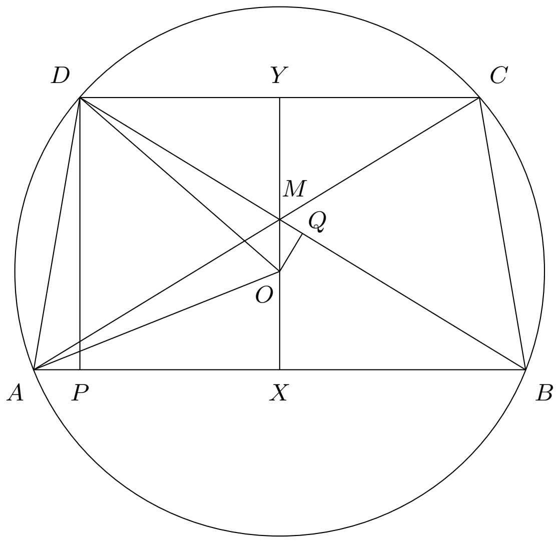
[Alternatively, applying Ptolemy's theorem to the cyclic quadrilateral $AOMD$ gives $AO \times MD + AD \times MO = AM \times OD$. Since $AO = AD = OD$, cancelling these gives $MD + MO = AM$.]

We know that X and Y are the midpoints of AB and CD respectively. From the 30-90-60 triangles AXM and DYM we have $AX = \frac{\sqrt{3}}{2}AM$ and $DY = \frac{\sqrt{3}}{2}DM$.
So $AB - CD = 2(\frac{\sqrt{3}}{2}AM - \frac{\sqrt{3}}{2}DM) = \sqrt{3}MO = 10\sqrt{3}$. $\boxed{1}$

Method 3

As in Method 2, $\triangle AOD$ is equilateral. $\boxed{1}$

Let P and Q be points on AB and BD respectively so that $DP \perp AB$ and $OQ \perp BD$.



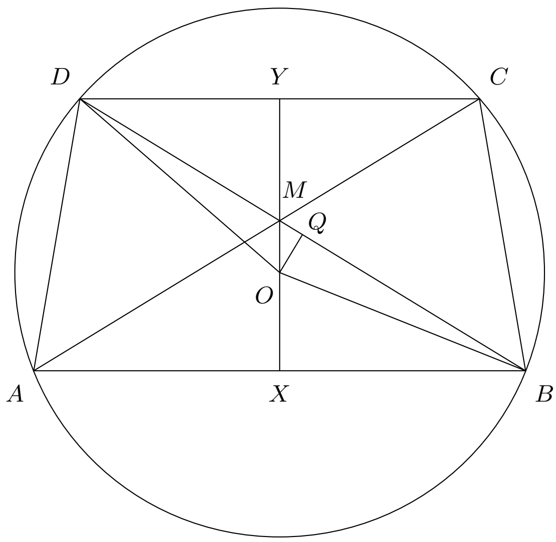
From the 30-60-90 triangle BDP, $DP = \frac{1}{2}BD$. Since $OB = OD$, triangles DOQ and BOQ are congruent. Hence $DQ = \frac{1}{2}DB = DP$. So triangles APD and OQD are congruent. Therefore $AP = OQ$. $\boxed{1}$

From the 30-60-90 triangle OMQ, $OQ = \frac{\sqrt{3}}{2}OM = 5\sqrt{3}$.
So $AB - CD = 2AX - 2DY = 2AX - 2PX = 2AP = 10\sqrt{3}$. $\boxed{1}$

Method 4

Let $x = BM$ and $y = DM$. From the 30-90-60 triangles BXM and DYM we have $BX = \frac{\sqrt{3}}{2}x$ and $DY = \frac{\sqrt{3}}{2}y$. Since X and Y are the midpoints of AB and CD respectively, $AB - CD = \sqrt{3}(x - y)$. $\boxed{1}$

Let Q be the point on BD so that $OQ \perp BD$.

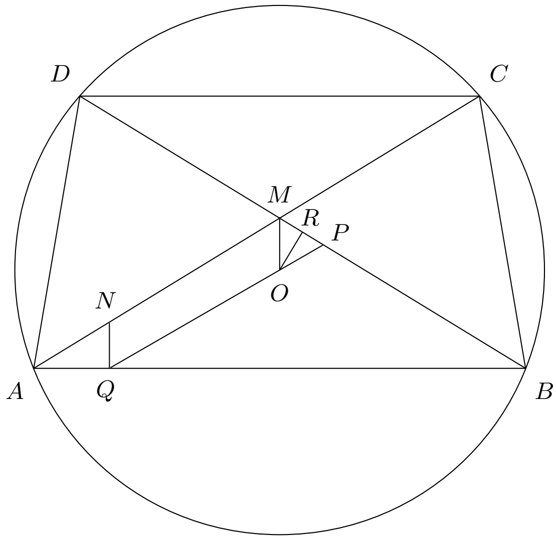


Since $BO = DO$, triangles BQO and DQO are congruent and $BQ = DQ$. Therefore $BQ = (x + y)/2$ and $MQ = x - BQ = (x - y)/2$. $\boxed{1}$

Since BXM is a 30-90-60 triangle, $\triangle OQM$ is also 30-90-60. Therefore $MQ = \frac{1}{2}MO = 5$. So $AB - CD = 2\sqrt{3}MQ = 10\sqrt{3}$. $\boxed{1}$

Method 5

We know that triangles AMB and DMC have the same angles. Let the line that passes through O and is parallel to AC intersect AB at Q and BD at P. Then $\angle BQP = \angle BAM$ and $\angle BPQ = \angle BMA$. So triangles BPQ and CMD are similar. $\boxed{1}$



Now $\angle QPD = \angle AMD = 60°$. So $\triangle OMP$ is equilateral. Let the line that passes through O and is perpendicular to BD intersect BD at R. Thus R bisects PM. Since $OD = OB$, triangles OBR and ODR are congruent and R bisects BD. Hence $DM = BP$ and triangles BPQ and CMD are congruent. So $AB - CD = AQ$. $\boxed{1}$

Draw QN parallel to OM with N on AM. Then $QN = OM = 10$ and $QN \perp AB$. So $\triangle ANQ$ is 30-60-90. Hence $AN = 20$ and, by Pythagoras, $AB - CD = \sqrt{400 - 100} = \sqrt{300} = 10\sqrt{3}$. $\boxed{1}$

Comment. Notice that $AB - CD$ is independent of the radius of the circumcircle $ABCD$. This is true for all cyclic trapeziums. If $\angle AMD = \alpha$, then by similar arguments to those above we can show that $AB - CD = 2MO \sin \alpha$.

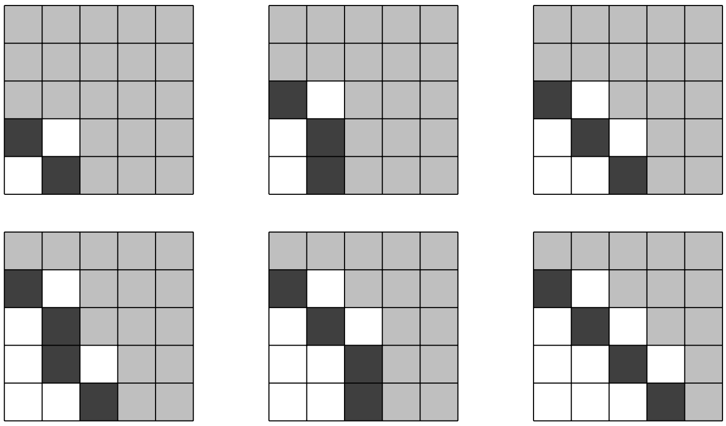
10. (a) Mark cells of the grid by coordinates, with (1, 1) being the cell in the lower-left corner of the grid. There are many ways of achieving a covering with exactly n black cells visible. Here's three.

Method 1

Putting each new tile *above* all previous tiles it overlaps with, place tiles in the following order with their lower-left cells on the listed grid cells:

(1, 1),
(1, 2), (2, 1),
(1, 3), (2, 2), (3, 1),
(1, 4), (2, 3), (3, 2), (4, 1),
and so on.

Continue this procedure to give black cells on the 'diagonal' just below the main diagonal and only white cells below. The following diagram shows this procedure for $n = 5$.



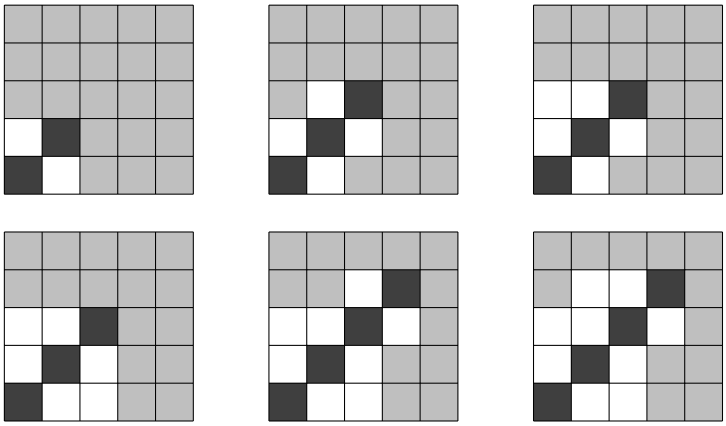
Start then in the upper-right corner and create black cells on the 'diagonal' just above the main diagonal and only white cells above. Finally put $n - 1$ tiles along the main diagonal. That will give n black cells on the main diagonal and white cells everywhere else. $\boxed{1}$

Method 2

Rotate all tiles so that the lower-left and upper-right cells are black. Putting each new tile *underneath* all previous tiles it overlaps with, place tiles in the following order with their lower-left cells on the listed grid cells:

(1, 1),
(2, 2), (1, 2), (2, 1),
(3, 3), (2, 3), (1, 3), (3, 2), (3, 1),
(4, 4), (3, 4), (2, 4), (1, 4), (4, 3), (4, 2), (4, 1),
and so on.

Continuing this procedure gives n black cells on the diagonal and white cells everywhere else. The following diagram shows this procedure for $n = 5$.



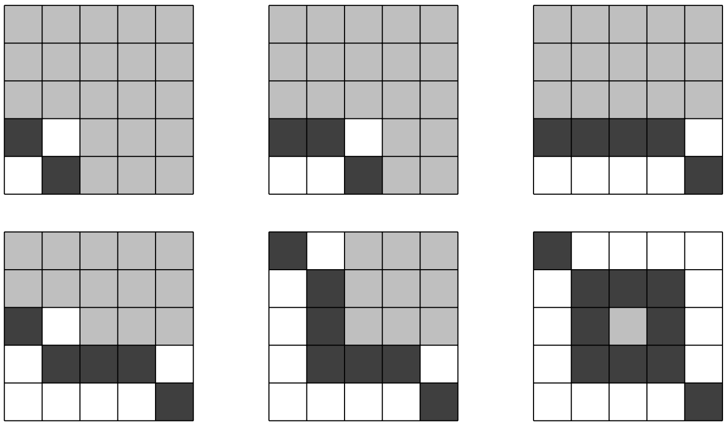
$\boxed{1}$

Method 3

Putting each new tile *above* all previous tiles it overlaps with, place tiles in the following order with their lower-left cells on the listed grid cells:

$(1, 1), (2, 1), (3, 1), \ldots, (n-1, 1),$
$(1, 2), (1, 2), (1, 3), \ldots, (1, n-1),$
$(n-1, n-1), (n-2, n-1), \ldots, (1, n-1),$
$(n-1, n-2), (n-1, n-3), \ldots, (n-1, 1),$

The following diagram shows this procedure for $n = 5$.



This gives a single border of all white cells except for black cells in the top-left and bottom-right corners of the grid. Now repeat this procedure for the inner $(n-2) \times (n-2)$ grid, then the inner $(n-4) \times (n-4)$ grid, and so on until an inner 1×1 or 2×2 grid remains. In both cases a single tile can cover the remaining uncovered grid cell(s) to produce a total covering that has n black cells on the diagonal and white cells everywhere else. $\boxed{1}$

(b) Suppose there is a covering of the $n \times n$ grid that has less than n black cells visible. Then there must be a row in which all visible cells are white. Any tile that overlaps this row has exactly two cells that coincide with cells in the row. These two cells are in the same row of the tile so one is white and one is black. Call these two cells a half-tile. Consider all half-tiles that cover cells in the row. Remove any half-tiles that have neither cell visible. The remaining half-tiles cover the row and all their visible cells are white. $\boxed{1}$

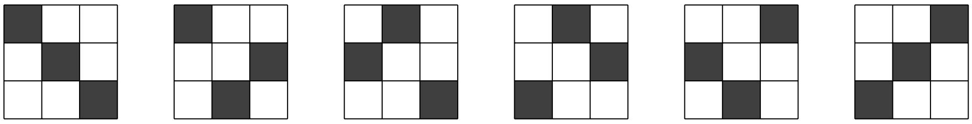
Consider any half-tile H_1. The black cell of H_1 must be covered by some half-tile H_2 and the white cell of H_1 must be visible. The black cell of H_2 must be covered by some half-tile H_3 and the white cell of H_2 must be visible. Thus we have a total of two visible white cells in the row. The black cell of H_3 must be covered by some half-tile H_4 and the white cell of H_3 must be visible. Thus we have a total of three visible white cells in the row.

So we may continue until we have a half-tile H_{n-1} plus a total of $n-2$ visible white cells in the row. The black cell of H_{n-1} must be covered by some half-tile H_n and the white cell of H_{n-1} must be visible. Thus we have a total of $n-1$ visible white cells in the row. As there are only n cells in the row, H_n must cover one of the visible white cells. This is a contradiction. So every covering of the $n \times n$ grid has at least n black cells visible. $\boxed{1}$

(c) From (a) and (b), the minimum number of visible black cells is n. From symmetry, the minimum number of visible white cells is n. Hence the maximum number of visible black cells is $n^2 - n$. $\boxed{1}$

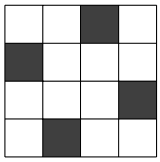
Investigation

(i) If a covering of a 3×3 grid has exactly 3 visible black cells, then the argument in Part (b) above shows that each row and each column must have exactly one visible black cell. The following diagram shows all possible patterns with exactly 3 black cells.



From symmetry we only need to consider the first two patterns. A covering to achieve the first pattern was given in Part (a) above. The second pattern can be achieved from the first by rotating a tile 90° and placing it in the bottom-right corner of the grid. bonus 1

(ii) The last tile to be placed shows two visible black cells and they share a vertex. However, in the following pattern no two black cells share a vertex.

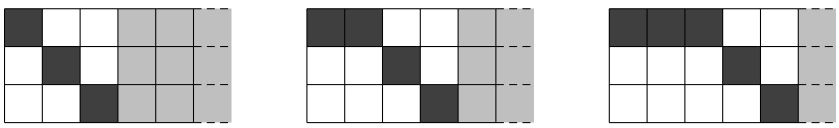


Thus not all patterns of 4 black cells and 12 white cells on a 4×4 grid can be achieved by a covering in which each new tile is placed on top of all previous tiles that it overlaps. bonus 1

Comment. This pattern can be achieved however if new tiles may be placed under previous tiles.

(iii) By the same argument as that in Part (b) above, the number of black cells exposed in any covering of the $n \times m$ grid is at least m.

We now show m is achievable. Number the columns 1 to m. Using the procedure in Part (a) Method 1 above, cover columns 1 to n to give n black cells on the main diagonal and white cells everywhere else. Now apply the same covering on columns 2 to $n+1$, then on columns 3 to $n+2$, and so on, finishing with columns $m-n+1$ to m. This procedure covers the entire $n \times m$ grid leaving exactly m black cells visible. The following diagram shows this procedure for $n = 3$.



So the minimum number of visible black cells in any covering of the $n \times m$ grid is m. From symmetry, the minimum number of visible white cells in any covering of the $n \times m$ grid is m. Hence the maximum number of visible black cells in any covering of the $n \times m$ grid is $nm - m = m(n-1)$. bonus 2

Marking Scheme

1.	A correct approach.	$\boxed{1}$
	Correct answer (15).	$\boxed{1}$
2.	A correct approach.	$\boxed{1}$
	Correct answer (168).	$\boxed{1}$
3.	A correct approach.	$\boxed{1}$
	Substantial progress.	$\boxed{1}$
	Correct answer (128).	$\boxed{1}$
4.	Three correct equations.	$\boxed{1}$
	Correct value of w.	$\boxed{1}$
	Correct answer (42).	$\boxed{1}$
5.	A correct approach.	$\boxed{1}$
	Further progress.	$\boxed{1}$
	Substantial progress.	$\boxed{1}$
	Correct answer (441).	$\boxed{1}$
6.	A correct approach.	$\boxed{1}$
	A useful equation.	$\boxed{1}$
	Substantial progress.	$\boxed{1}$
	Correct answer (28).	$\boxed{1}$
7.	A correct approach.	$\boxed{1}$
	Further progress.	$\boxed{1}$
	Substantial progress.	$\boxed{1}$
	Correct answer (927).	$\boxed{1}$
8.	A correct approach.	$\boxed{1}$
	Further progress.	$\boxed{1}$
	Substantial progress.	$\boxed{1}$
	Correct answer (625).	$\boxed{1}$
9.	Establishing triangles AMB and CMD are isosceles.	$\boxed{1}$
	Establishing angles AMO and BMO are $60°$.	$\boxed{1}$
	Further progress.	$\boxed{1}$
	Substantial progress.	$\boxed{1}$
	Correct answer ($10\sqrt{3}$).	$\boxed{1}$
10.	(a) A correct covering with exactly n black cells visible.	$\boxed{1}$
	(b) Correct proof that at least n black cells will be visible.	$\boxed{2}$
	(c) Correct answer ($n^2 - n$) and proof.	$\boxed{1}$
	Investigation:	
	(i) Correct coverings for all patterns.	$\boxed{\text{bonus 1}}$
	(ii) A convincing explanation.	$\boxed{\text{bonus 1}}$
	(iii) Correct answer ($m(n-1)$) and proof.	$\boxed{\text{bonus 2}}$

The Mathematics Olympiads are supported by the Australian Government
Department of Education through the Mathematics and Science Participation Program.

BẢNG PHÂN TÍCH KẾT QUẢ CHI TIẾT VÀ DANH SÁCH HỌC SINH ĐẠT CHỨNG NHẬN ``GIẢI THƯỞNG’’ VÀ ‘‘XUẤT SẮC’’

NĂM 2014

AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD STATISTICS

DISTRIBUTION OF AWARDS/SCHOOL YEAR

YEAR	NUMBER OF STUDENTS	NUMBER OF AWARDS				
		Prize	High Distinction	Distinction	Credit	Participation
8	336	2	13	35	97	189
9	390	5	32	62	106	185
10	413	14	42	70	124	163
Other	167	1	9	12	39	106
Total	1306	22	96	179	366	643

NUMBER OF CORRECT ANSWERS QUESTIONS 1–8

YEAR	NUMBER CORRECT/QUESTION							
	1	2	3	4	5	6	7	8
8	132	165	225	157	84	24	10	87
9	173	231	270	221	119	68	21	122
10	209	268	277	226	149	87	23	147
Other	64	79	115	72	37	12	2	38
Total	578	743	887	676	389	191	56	394

MEAN SCORE/QUESTION/SCHOOL YEAR

YEAR	NUMBER OF STUDENTS	MEAN SCORE			OVERALL MEAN
		Question			
		1–8	9	10	
8	336	8.2	0.1	0.1	8.5
9	390	9.8	0.5	0.2	10.5
10	413	10.7	0.6	0.3	11.6
Other	167	7.7	0.2	0.1	7.9
All Years	1306	9.4	0.4	0.2	10.0

AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD RESULTS

NAME	SCHOOL	YEAR	TOTAL	AWARD
Yong See Foo	Nossal High School VIC	10	35	Prize
Yu Tse Lee	Raffles Institution SNG	10	35	Prize
Yuan Lee	Raffles Institution SNG	10	35	Prize
Seyoon Ragavan	Knox Grammar School NSW	10	35	Prize
Jianzhi Wang	Raffles Institution SNG	8	35	Prize
Peng Jun Bryan Wang	Hwa Chong Institution SNG	9	35	Prize
Yang Gan	Raffles Institution SNG	10	34	Prize
Ilia Kucherov	Westall Secondary College VIC	10	34	Prize
Glen Wei An Lim	Raffles Institution SNG	10	34	Prize
Zhe Hui Lim	Hwa Chong Institution SNG	10	34	Prize
Yijia Liu	Raffles Institution SNG	10	34	Prize
Jerry Mao	Caulfield Grammar School Wheelers Hill VIC	8	34	Prize
Sheldon Kieren Tan	Raffles Institution SNG	10	34	Prize
Yikai Wu	Hwa Chong Institution SNG	10	34	Prize
Ma Zhao Yu	Raffles Institution SNG	9	34	Prize
Jongmin Lim	Killara High School NSW	10	33	Prize
Pengfei Zhao	Hwa Chong Institution SNG	10	33	Prize
Yaxuan Zheng	Raffles Girls' School (Secondary) SNG	9	33	Prize
Eryuan Sheng	Newington College NSW	10	32	Prize
Austin Zhang	Sydney Grammar School NSW	9	32	Prize
Shengwei Lu	Hwa Chong Institution SNG	9	31	Prize
Kevin Xian	James Ruse Agricultural High School NSW	10	31	Prize
Linus Cooper	James Ruse Agricultural High School NSW	8	30	High Distinction
Yu Fu	Anglo-Chinese School (Independent) SNG	10	30	High Distinction
Ziming Xue	Anglo-Chinese School (Independent) SNG	10	30	High Distinction
Hong Pei Goh	Hwa Chong Institution SNG	9	29	High Distinction
Evgeni Kayryakov	Childrens Academy 21st Century BUL	7	29	High Distinction

Steven Lim	Hurlstone Agricultural High School NSW	8	29	High Distinction
Jack Liu	Brighton Grammar School VIC	8	29	High Distinction
Hristo Papazov	Childrens Academy 21st Century BUL	9	29	High Distinction
Likai Tan	Raffles Institution SNG	9	29	High Distinction
Yanlong Wu	Hwa Chong Institution SNG	10	29	High Distinction
Ariel Pratama Junaidi	Anglo-Chinese School (Independent) SNG	9	28	High Distinction
Meng Liao	Anglo-Chinese School (Independent) SNG	9	28	High Distinction
Wen Zhang	St Joseph's College, Gregory Terrace QLD	8	28	High Distinction
Puhua Cheng	Raffles Institution SNG	7	27	High Distinction
Jun Kim	Trinity Grammar School VIC	10	27	High Distinction
Shang Hui Koh	Hwa Chong Institution SNG	9	27	High Distinction
Winfred Kong	Hwa Chong Institution SNG	9	27	High Distinction
Tianyi Liu	Raffles Institution SNG	10	27	High Distinction
Alex Lugovskoy	Willetton Senior High School WA	10	27	High Distinction
Michael Robertson	Campbell High School ACT	10	27	High Distinction
Jun Hao Tin	Hwa Chong Institution SNG	10	27	High Distinction
Letian Yu	Hwa Chong Institution SNG	10	27	High Distinction
Wilson Zhao	Killara High School NSW	10	27	High Distinction
Thomas Baker	Scotch College VIC	10	26	High Distinction
Siew Keng Hun	Raffles Institution SNG	10	26	High Distinction
Adrian Law	James Ruse Agricultural High School NSW	9	26	High Distinction
Zlatina Mileva	Childrens Academy 21st Century BUL	7	26	High Distinction
Madeline Nurcombe	Cannon Hill Anglican College QLD	10	26	High Distinction
Georgi Rusinov	Childrens Academy 21st Century BUL	10	26	High Distinction
Simon Yang	James Ruse Agricultural High School NSW	10	26	High Distinction
Jiaqi Bao	Hwa Chong Institution SNG	9	25	High Distinction
Xuhui Chen	Hwa Chong Institution SNG	9	25	High Distinction
Devin He	Christ Church Grammar School WA	10	25	High Distinction

Caleb Yong Quan Leow	Raffles Institution SNG	8	25	High Distinction
Jing Qian	Hwa Chong Institution SNG	10	25	High Distinction
Shuwei Wang	Hwa Chong Institution SNG	9	25	High Distinction
Khor Jun Wei	Raffles Institution SNG	7	25	High Distinction
Zhiqiu Yu	Anglo-Chinese School (Independent) SNG	9	25	High Distinction
William Hu	Christ Church Grammar School WA	8	24	High Distinction
Shivasankaran Jayabalan	Rossmoyne Senior High School WA	8	24	High Distinction
Tony Jiang	Scotch College VIC	9	24	High Distinction
Yong Le Isaac Lee	Raffles Institution SNG	8	24	High Distinction
Oswald Li	Scotch College VIC	10	24	High Distinction
Yingtong Li	Pembroke School SA	10	24	High Distinction
James Manton-Hall	Sydney Grammar School NSW	10	24	High Distinction
Anthony Pisani	St Paul's Anglican Grammar School VIC	7	24	High Distinction
Jiaqi Wu	Anglo-Chinese School (Independent) SNG	10	24	High Distinction
Anand Bharadwaj	Trinity Grammar School VIC	8	23	High Distinction
Hu Chen	The King's School NSW	9	23	High Distinction
Tianxiao Chen	Hwa Chong Institution SNG	9	23	High Distinction
Bobby Dey	James Ruse Agricultural High School NSW	9	23	High Distinction
Seah Fengyu	Raffles Institution SNG	8	23	High Distinction
Rachel Hauenschild	Kenmore State High School QLD	9	23	High Distinction
Scarlett He	All Saints Anglican Senior School QLD	10	23	High Distinction
Colin Huang	North Sydney Boys High School NSW	10	23	High Distinction
Xianyi Huang	Baulkham Hills High School NSW	9	23	High Distinction
Vladimira Irincheva	Childrens Academy 21st Century BUL	7	23	High Distinction
Anqi Li	Raffles Girls' School (Secondary) SNG	2	23	High Distinction
Luozhiyu Lin	Anglo-Chinese School (Independent) SNG	9	23	High Distinction

Steven Liu	James Ruse Agricultural High School NSW	9	23	High Distinction
Isabel Longbottom	Rossmoyne Senior High School WA	9	23	High Distinction
Ivo Petrov	Childrens Academy of Sciences, Arts and Sports BUL	7	23	High Distinction
Nicholas Pizzino	Christ Church Grammar School WA	10	23	High Distinction
Virinchi Rallabhandi	Perth Modern School WA	10	23	High Distinction
Kohsuke Sato	Christ Church Grammar School WA	9	23	High Distinction
Kirill Saylov	Brisbane Grammar School QLD	9	23	High Distinction
Senan Sekhon	Anglo-Chinese School (Independent) SNG	9	23	High Distinction
Katrina Shen	James Ruse Agricultural High School NSW	7	23	High Distinction
Hang Sheng	Rossmoyne Senior High School WA	10	23	High Distinction
Rohith Srinivas	Raffles Institution SNG	8	23	High Distinction
Eric Tan	James Ruse Agricultural High School NSW	10	23	High Distinction
Yin Tang	Hwa Chong Institution SNG	9	23	High Distinction
Bella Tao	Presbyterian Ladies' College VIC	10	23	High Distinction
Alexander Barber	Scotch College VIC	10	22	High Distinction
Brian Chau	Sydney Grammar School NSW	10	22	High Distinction
Michelle Chen	Methodist Ladies' College VIC	10	22	High Distinction
Zixuan Chen	Caulfield Grammar School (Caulfield Campus) VIC	8	22	High Distinction
Alan Cheng	Perth Modern School WA	10	22	High Distinction
Aniruddh Chennapragada	James Ruse Agricultural High School NSW	8	22	High Distinction
Clement Chiu	The King's School NSW	10	22	High Distinction
Bedanta Dhal	Perth Modern School WA	10	22	High Distinction
Richard Gong	Sydney Grammar School NSW	9	22	High Distinction
Tasneef Helal	James Ruse Agricultural High School NSW	9	22	High Distinction
Cameron Hinton	James Ruse Agricultural High School NSW	10	22	High Distinction
Gideon Kharistia	Anglo-Chinese School (Independent) SNG	9	22	High Distinction

Michelle Kim	James Ruse Agricultural High School NSW	10	22	High Distinction
Aaron Lawrence	James Ruse Agricultural High School NSW	10	22	High Distinction
Clifford Lee	Wesley College WA	10	22	High Distinction
Zixuan Lu	Hwa Chong Institution SNG	10	22	High Distinction
Oliver McLeish	Scotch College VIC	9	22	High Distinction
Daniel Qin	Scotch College VIC	9	22	High Distinction
Alexander Rohl	Perth Modern School WA	10	22	High Distinction
Dibyendu Roy	Sydney Boys High School NSW	9	22	High Distinction
William Song	Scotch College VIC	10	22	High Distinction
Victor Wu	Trinity Grammar School NSW	10	22	High Distinction
Zhen Wu	Anglo-Chinese School (Independent) SNG	9	22	High Distinction

ĐỀ THI VÀ ĐÁP ÁN CHI TIẾT

CUỘC THI VÔ ĐỊCH TOÁN CẤP TRUNG HỌC ÚC MỞ RỘNG NĂM 2013

AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

Australian Intermediate Mathematics Olympiad 2013

Questions

1. Find the area in cm^2 of a rhombus whose side length is 29 cm and whose diagonals differ in length by 2 cm.

[**2 marks**]

2. How many 4-digit numbers are there whose digit product is 60?

[**2 marks**]

3. A base 7 three-digit number has its digits reversed when written in base 9. Find the decimal representation of the number.

[**3 marks**]

4. The prime numbers p, q, r satisfy the simultaneous equations $pq + pr = 80$ and $pq + qr = 425$. Find the value of $p + q + r$.

[**3 marks**]

5. How many pairs of 3-digit palindromes are there such that when they are added together, the result is a 4-digit palindrome? For example, $232 + 989 = 1221$ gives one such pair.

[**4 marks**]

6. ABC is an equilateral triangle with side length $2013\sqrt{3}$. Find the largest diameter for a circle in one of the regions between $\triangle ABC$ and its inscribed circle.

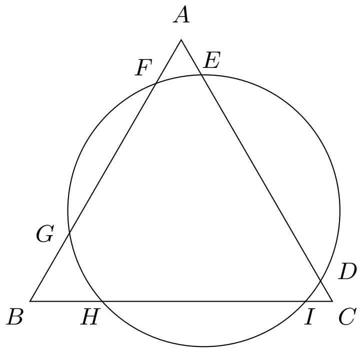
[**4 marks**]

7. If a, b, c, d are positive integers with sum 63, what is the maximum value of $ab + bc + cd$?

[**4 marks**]

PLEASE TURN OVER THE PAGE FOR QUESTIONS 8, 9 AND 10

8. A circle meets the sides of an equilateral triangle ABC at six points D, E, F, G, H, I as shown. If $AE = 4$, $ED = 26$, $DC = 2$, $FG = 14$, and the circle with diameter HI has area πb, find b.



[**4 marks**]

9. A box contains some identical tennis balls. The ratio of the total volume of the tennis balls to the volume of empty space surrounding them in the box is $1 : k$, where k is an integer greater than 1.

A prime number of tennis balls is removed from the box. The ratio of the total volume of the remaining tennis balls to the volume of empty space surrounding them in the box is $1 : k^2$. Find the number of tennis balls that were originally in the box.

[**5 marks**]

10. I have a $1\,\mathrm{m} \times 1\,\mathrm{m}$ square, which I want to cover with three circular discs of equal size (which are allowed to overlap). Show that this is possible if the discs have diameter $1008\,\mathrm{mm}$.

[**4 marks**]

Investigation

Two discs of equal diameter cover a $1\,\mathrm{m} \times 1\,\mathrm{m}$ square. Find their minimum diameter.

[**3 bonus marks**]

The Mathematics Olympiads are supported by the Australian Government
Department of Education, Employment and Workplace Relations
through the Mathematics and Science Participation Program.

Australian Mathematical Olympiad Committee

A department of the Australian Mathematics Trust

Australian Intermediate Mathematics Olympiad 2013

Solutions

1. *Method 1*

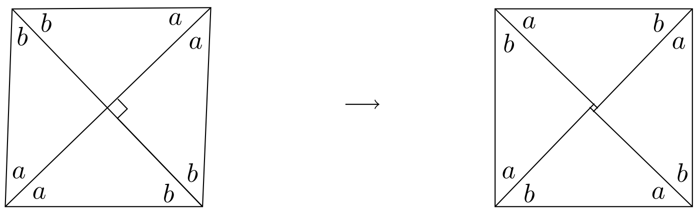
Let the diagonals be $2x$ and $2x+2$. The diagonals of a rhombus bisect each other at right angles. Hence they partition the rhombus into four congruent right-angled triangles each with hypotenuse 29. Thus the area of the rhombus is $4 \times \frac{1}{2}x(x+1) = 2x^2 + 2x$. $\boxed{1}$

From Pythagoras, $x^2 + (x+1)^2 = 29^2 = 841$. So $2x^2 + 2x = \mathbf{840}$. $\boxed{1}$

Method 2

The diagonals of a rhombus bisect each other at right angles. Hence they partition the rhombus into four congruent right-angled triangles each with hypotenuse 29 and short sides differing by one. $\boxed{1}$

Because the small angles of these triangles are complementary, the triangles can be rearranged to form a square of side 29 with a unit square hole.



Thus the area of the rhombus is $29^2 - 1 = \mathbf{840}$. $\boxed{1}$

2. Since $60 = 2^2 \times 3 \times 5$, only the digits 1, 2, 3, 4, 5 and 6 can be used. The only combinations of four of these digits whose product is 60 are (1, 2, 5, 6), (1, 3, 4, 5), (2, 2, 3, 5). $\boxed{1}$

There are 24 ways to arrange four different digits and 12 ways to arrange four digits of which two are the same. So the total number of required 4-digit numbers is $24 + 24 + 12 = \mathbf{60}$. $\boxed{1}$

3. Call the number $(abc)_7$, where a, b, c are digits less than 7. Then $(abc)_7 = (cba)_9$. $\boxed{1}$

Hence $49a + 7b + c = 81c + 9b + a$.
So $48a = 80c + 2b$ or $24a = 40c + b$.
Then $b = 24a - 40c = 8(3a - 5c)$.
Since b is both a digit less than 7 and a multiple of 8, it must be 0. $\boxed{1}$

Now we have $3a - 5c = 0$, or $3a = 5c$.
Since a and c are digits less than 7, $a = 5$ and $c = 3$.
So the number is $(503)_7 = 5 \times 49 + 3 \times 1 = \mathbf{248}$. $\boxed{1}$

4. *Method 1*

We have $p(q+r) = 80 = 2^4 \times 5$, $q(p+r) = 425 = 5^2 \times 17$, $r(q-p) = 345 = 3 \times 5 \times 23$. $\boxed{1}$

So $p = 2$ or 5, $q = 5$ or 17, $r = 3$, 5, or 23.
From $p(q+r) = 80$, if $p = 5$ then $q + r = 16$, which has no solution. $\boxed{1}$

So $p = 2$, then $q + r = 40$, hence $q = 17$ and $r = 23$.
Therefore $p + q + r$ is $2 + 17 + 23 = \mathbf{42}$. $\boxed{1}$

Method 2

We have $p(q+r) = 80 = 2^4 \times 5$ and $q(p+r) = 425 = 5^2 \times 17$. $\boxed{1}$

So $p = 2$ or 5 and $q = 5$ or 17.
From $q(p+r) = 425$, if $q = 5$ then $p + r = 85$.
So $r = 83$ or 80, which both contradict $p(q+r) = 80$. $\boxed{1}$

Hence $q = 17$ and $p + r = 25$. Since r is prime, $p = 2$ and $r = 23$.
Therefore $p + q + r$ is $2 + 17 + 23 = \mathbf{42}$. $\boxed{1}$

5. Write the sum as follows:

$$\begin{array}{r} aba \\ \underline{cdc} \\ \underline{effe} \end{array}$$

It is clear from the 1000s column that $e = 1$. $\boxed{1}$

So from the units column, $a + c = 1$ or $a + c = 11$.
But the carry from the 100s column means that $a + c = 11$. $\boxed{1}$

Method 1

Without loss of generality, take $a < c$.
So the possible solution pairs for (a, c) are: (2, 9), (3, 8), (4, 7), (5, 6).
Now the 10s column gives $b + d + 1 = f$ or $b + d + 1 = f + 10$.

Case 1: $b + d + 1 = f$.

Since there is no carry to the 100s column and $a + c = 11$, we have $f = 1$.
Thus $b + d = 0$, hence $b = d = 0$.
So this gives four solutions:
$202 + 909 = 1111$, $303 + 808 = 1111$, $404 + 707 = 1111$, $505 + 606 = 1111$. $\boxed{1}$

Case 2: $b + d + 1 = f + 10$.

Since there is a carry to the 100s column, $a + c + 1 = 12$.
Then $f = 2$, hence $b + d = 11$.
For each pair of values of a and c, there are then eight solution pairs for (b, d):
(2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3), (9, 2).
So this gives $4 \times 8 = 32$ solutions ($222 + 999 = 1221$, $232 + 989 = 1221$, etc).

Hence the number of solution pairs overall is $4 + 32 = \mathbf{36}$. $\boxed{1}$

Method 2

We know that $e = 1$, $a + c = 11$, and the carry from the 10s column is at most 1.
Hence $f = 1$ or 2. Therefore $b + d = 0$ or 11 respectively. $\boxed{1}$

For each pair of values of a and c, there are then nine solution pairs for (b, d):
(0, 0), (2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3), (9, 2).
Thus the number of required pairs of 3-digit palindromes is $4 \times 9 = \mathbf{36}$. $\boxed{1}$

6. *Method 1*

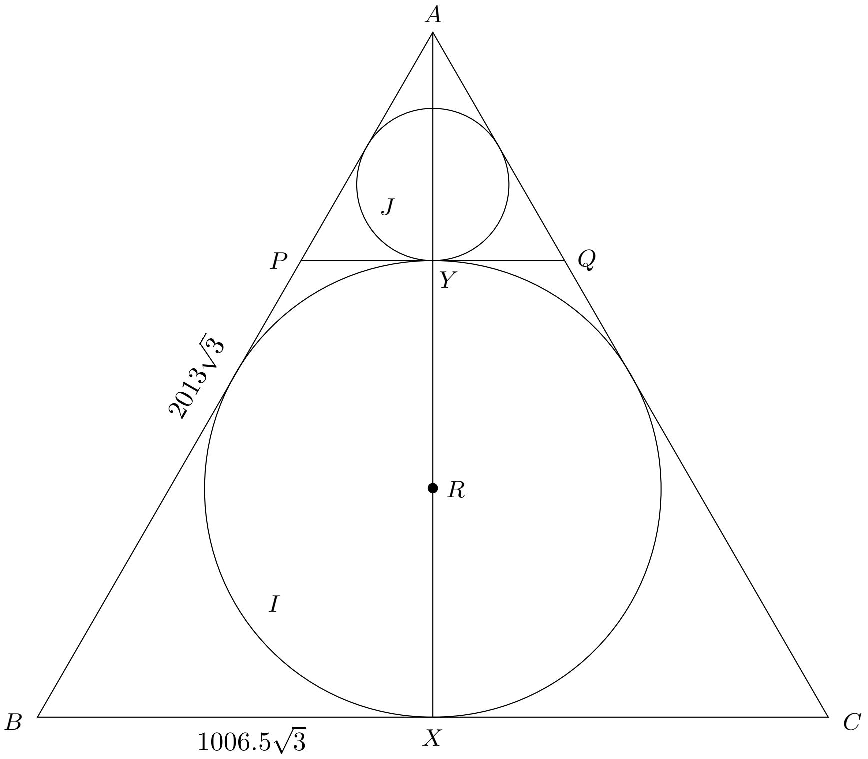
Let I be the incircle of $\triangle ABC$ and let J be the largest circle in the top region between $\triangle ABC$ and I.

Let R be the incentre of $\triangle ABC$. Then AR bisects $\angle BAC$. Extend AR to meet BC at X. Since $\triangle ABC$ is equilateral, X is the midpoint of BC. By symmetry, R lies on all medians of $\triangle ABC$. Hence $RX = \frac{1}{3}AX$. AX is also perpendicular to BC. $\boxed{1}$

Since J touches AB and AC, its centre is also on AX. Hence I and J touch at some point Y on AX. Let their common tangent meet AB at P and AC at Q. Then PQ and BC are parallel. Hence $\triangle APQ$ is similar to $\triangle ABC$. $\boxed{1}$

So $\triangle APQ$ is equilateral and its altitude $AY = AX - YX = AX - \frac{2}{3}AX = \frac{1}{3}AX$. Since J is the incircle of $\triangle APQ$, its radius is $\frac{1}{3}AY = \frac{1}{9}AX$. Since $\triangle ABX$ is 30-60-90, $AX = \sqrt{3} \times 1006.5\sqrt{3} = 3 \times 1006.5$. So the diameter of $J = 2 \times \frac{1}{9}AX = 2 \times 1006.5/3 = \mathbf{671}$.

$\boxed{1}$



$\boxed{1}$

Method 2

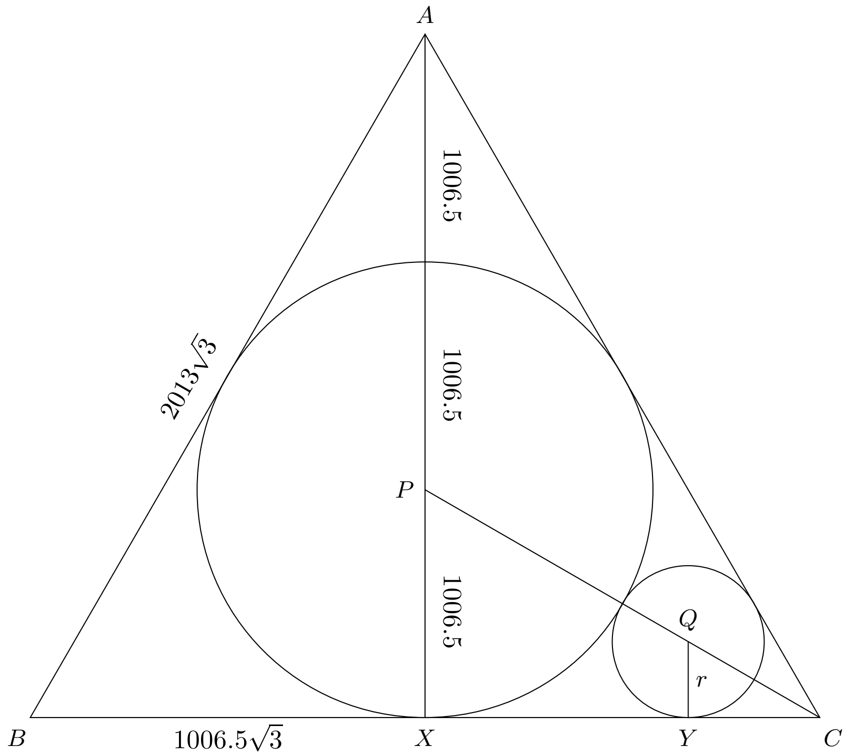
Let r be the radius of the smaller circle.

Let P be the incentre of $\triangle ABC$. Then AP bisects $\angle BAC$ and CP bisects $\angle ACB$. Extend AP to meet BC at X. Since $\triangle ABC$ is equilateral, X is the midpoint of BC, AX is perpendicular to BC, and $PA = PC$.

Thus $\triangle ABX$ and $\triangle CPX$ are 30-60-90. Hence $AX = \sqrt{3} \times 1006.5\sqrt{3} = 3 \times 1006.5$ and $PX = XC/\sqrt{3} = 1006.5$. So $AP = 2PX$. $\boxed{1}$

Since the smaller circle touches AC and BC, its centre, Q, lies on CP. Let Y be the point where the smaller circle touches BC. Then QY is perpendicular to BC. Hence $\triangle CYQ$ is similar to $\triangle CXP$. $\boxed{1}$

So $\dfrac{r}{XP} = \dfrac{CQ}{CP} = \dfrac{CQ}{AP} = \dfrac{CQ}{2PX}$.
Hence $2r = CQ = CP - QP = AP - QP = 2 \times 1006.5 - (1006.5 + r)$.
Therefore $3r = 1006.5$ and $2r = 2013/3 = \mathbf{671}$. $\boxed{1}$



$\boxed{1}$

7. *Method 1*

For all real numbers x and y, we have $(x-y)^2 \geq 0$. So $x^2+y^2 \geq 2xy$, hence $(x+y)^2 \geq 4xy$ and $xy \leq (x+y)^2/4$. $\boxed{1}$

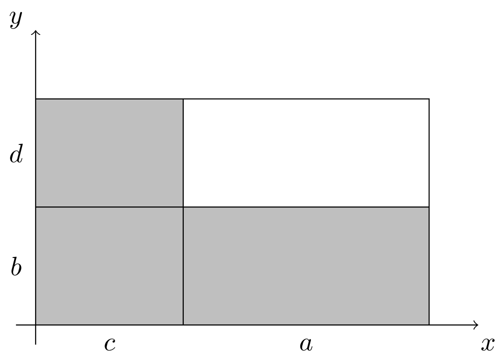
Letting $x = a+c$ and $y = b+d$ gives $(a+c)(b+d) \leq (a+b+c+d)^2/4$.
So $ab+bc+cd+da \leq 63^2/4 = 3969/4 = 992.25$. $\boxed{1}$

Since a, b, c, d are positive integers, the last inequality can be written as $ab+bc+cd+da \leq 992$. Hence $ab+bc+cd \leq 992 - da \leq 991$. $\boxed{1}$

It remains to show that 991 is achievable. Suppose $ab+bc+cd = 991$ and $a = d = 1$. Then $(1+b)(1+c) = 992 = 2^5 \times 31$. So $b = 30$ and $c = 31$ is a solution. Thus the maximum value of $ab+bc+cd$ is **991**. $\boxed{1}$

Method 2

Consider the rectangles $a \times b$, $b \times c$, $c \times d$, $a \times d$ arranged as follows. We wish to maximise the shaded area.



$\boxed{1}$

Let $u = a+c$ and $v = b+d$. For fixed u and v, the shaded area is maximum when $a = d = 1$. So, to maximise the shaded area, we need to maximise the area of the $u \times v$ rectangle with $u = c+1$, $v = b+1$, and $u+v = 63$. $\boxed{1}$

The area of the $u \times v$ rectangle is $uv = u(63-u)$. The graph of $y = x(63-x)$ is a parabola with its maximum at $x = 63/2$. Hence the maximum value of uv is attained when u is as close as possible to $63/2$. Thus $u = 31$ and $v = 32$ or vice versa. $\boxed{1}$

So the maximum shaded area is $31 \times 32 - 1 =$ **991**. $\boxed{1}$

Method 3

From symmetry we may assume $b \leq c$.

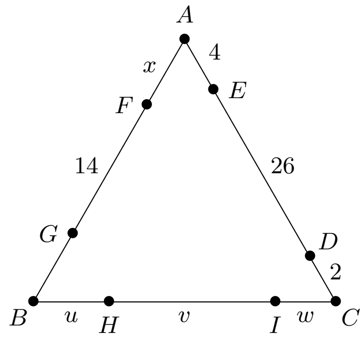
Then $ab+bc+cd \leq ac+bc+cd = c(a+b+d) = c(63-c)$. $\boxed{1}$

The graph of $y = x(63-x)$ is a parabola with its maximum at $x = 63/2$. Hence the maximum value of $c(63-c)$ is attained when c is as close as possible to $63/2$. Thus $c = 31$ or 32. $\boxed{1}$

If $b = c$, then $a+b+c+d \geq 1+31+31+1 = 64$, a contradiction. So $b < c$ and we have $ab+bc+cd < (31)(32) = 992$. $\boxed{1}$

It remains to show that 991 is achievable. If $c = 31$, $b = 30$, and $a = d = 1$, then $ab+bc+cd = 991$. Thus the maximum value of $ab+bc+cd$ is **991**. $\boxed{1}$

8. Let $BH = u$, $HI = v$, $IC = w$, and $AF = x$.



The intersecting secant theorem at A gives $4 \times 30 = x(x + 14)$.
Hence $x^2 + 14x - 120 = 0$. So $(x + 20)(x - 6) = 0$ and $x = 6$. Therefore $GB = 12$. $\boxed{1}$

The intersecting secant theorem at B gives $12 \times 26 = u(u + v) = u(32 - w)$. (1)
The intersecting secant theorem at C gives $2 \times 28 = w(w + v) = w(32 - u)$. (2) $\boxed{1}$

Subtracting (2) from (1) gives:

$$\begin{aligned} 256 &= 32(u - w) \\ u &= w + 256/32 = w + 8 \\ v &= 32 - u - w = 24 - 2w \end{aligned}$$ $\boxed{1}$

Substituting in (2) gives:

$$\begin{aligned} 56 &= w(24 - w) \\ w^2 - 24w + 56 &= 0 \\ w &= \left(24 \pm \sqrt{24^2 - 224}\right)/2 \\ v &= 4\sqrt{22} \end{aligned}$$

Hence $\pi b = \pi(2\sqrt{22})^2 = \pi 88$ and $b = \mathbf{88}$. $\boxed{1}$

9. Let the volume of the box be V_B and the volume of a single tennis ball be V_T.
Suppose there are N tennis balls to begin.
The total volume of the tennis balls is NV_T and the volume of empty space surrounding them is $V_B - NV_T$.
From the ratio given, $V_B - NV_T = kNV_T$, so $V_B = NV_T + kNV_T = (k + 1)NV_T$. $\boxed{1}$

Let P be the number of balls removed, where P is a prime.
The total volume of the remaining tennis balls is $(N - P)V_T$ and the volume of empty space surrounding them is
$V_B - (N - P)V_T = (k + 1)NV_T - (N - P)V_T = kNV_T + PV_T = (kN + P)V_T$. $\boxed{1}$

From the ratio given, $(kN + P)V_T = k^2(N - P)V_T$.
So $kN + P = k^2N - k^2P$, hence $N = P(k^2 + 1)/(k^2 - k)$. $\boxed{1}$

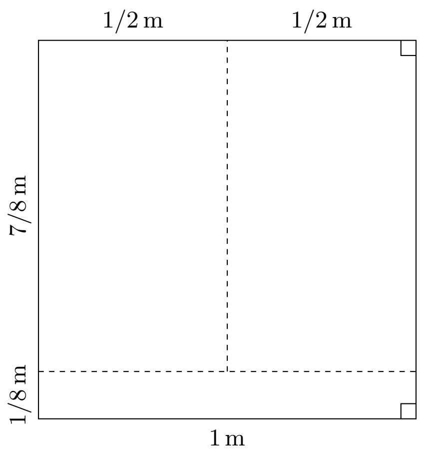
Now N is an integer and k and $k^2 + 1$ are relatively prime, so k divides P.
But P is a prime and $k > 1$ so $k = P$.
Thus $N = (P^2 + 1)/(P - 1) = P + 1 + 2/(P - 1)$. $\boxed{1}$

Since N is an integer, $P - 1$ divides 2.
So $P = 2$ or $P = 3$. Either way $N = 5$. Thus the number of tennis balls originally in the box is **5**. $\boxed{1}$

Comment. The algebra can be simplified by rescaling to let $V_T = 1$.

10. *Method 1*

Subdivide the square into three rectangles: one measuring $1\,\mathrm{m} \times \frac{1}{8}\,\mathrm{m}$ and each of the other two measuring $\frac{1}{2}\,\mathrm{m} \times \frac{7}{8}\,\mathrm{m}$.



$\boxed{1}$

Note that $\sqrt{1+(\frac{1}{8})^2} = \sqrt{\frac{65}{64}}$ and $\sqrt{(\frac{1}{2})^2+(\frac{7}{8})^2} = \sqrt{\frac{65}{64}}$.

So each of the three rectangles have diagonals of length $\sqrt{\frac{65}{64}}$ and can therefore be covered by a disc with this diameter. $\boxed{1}$

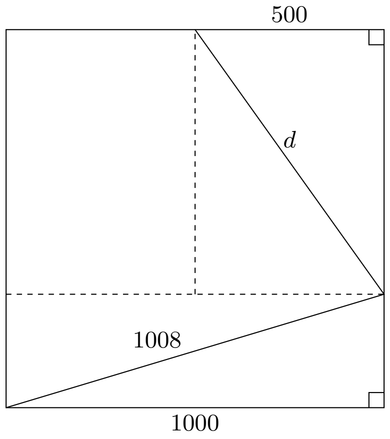
Now $\sqrt{\frac{65}{64}} < 1.008 \Leftrightarrow 65 < 64(1.008)^2$ and

$64(1+0.008)^2 > 64 \times 1.016 = 64(1+0.01+0.006) = 64+0.64+0.384 = 65.024 > 65.$ $\boxed{1}$

So $\sqrt{\frac{65}{64}} < 1.008$. Therefore it is possible for three discs each with diameter 1008 mm to cover the square. $\boxed{1}$

Method 2

Construct two right-angled triangles in the $1\,\text{m} \times 1\,\text{m}$ square as shown.



We need to show $d < 1008$. $\boxed{1}$

$$\begin{aligned}\text{Pythagoras gives } d^2 &= 500^2 + \left(1000 - \sqrt{1008^2 - 1000^2}\right)^2 \\ &= 500^2 + \left(1000 - \sqrt{(1008-1000)(1008+1000)}\right)^2 \\ &= 500^2 + \left(1000 - \sqrt{8 \times 2008}\right)^2 \\ &= 250000 + 1000000 + 16064 - 2000\sqrt{16064}.\end{aligned}$$ $\boxed{1}$

So $d^2 - 1008^2 = d^2 - 1016064 = 250000 - 2000\sqrt{16064}$. $\boxed{1}$

Hence $d < 1008 \Leftrightarrow 125^2 < 16064$, which is true since $125^2 = 15625$. $\boxed{1}$

Investigation

The minimum diameter is $500\sqrt{5} \approx 1118\,\text{mm}$.

Divide the square into two rectangles each $1000\,\text{mm} \times 500\,\text{mm}$. The length of the diagonal of each rectangle is $\sqrt{1000^2 + 500^2} = 500\sqrt{5}\,\text{mm}$. Hence the square can be covered by two discs of this diameter. $\boxed{\text{bonus } 1}$

Suppose the square can be covered by two discs of shorter diameter. One of the discs must cover at least two of the vertices of the square. If two of these vertices were diagonally opposite on the square, then the diameter of the disc would be at least the length of the diagonal of the square, which is approximately 1400 mm. So the disc covers exactly two vertices of the square and they are on the same side of the square. $\boxed{\text{bonus } 1}$

Denote the square by $ABCD$. We may assume the disc covers A and B, and intersects AD at a point X. The disc covers BX and its diameter is less than $500\sqrt{5}$. Hence BX is less than $500\sqrt{5}$, so $AX^2 < (500\sqrt{5})^2 - 1000^2 = (125 - 100)10000 = 250000$ and $AX < 500$. The second disc must cover points X, C, and D. Since $XD > 500$, $XC > 500\sqrt{5}$. Then the diameter of the second disc is greater than $500\sqrt{5}$, a contradiction.

So the diameters of the two covering discs cannot be less than $500\sqrt{5}$. $\boxed{\text{bonus } 1}$

Marking Scheme

1. A correct approach. $\boxed{1}$
Correct answer (840). $\boxed{1}$

2. Three correct combinations of digits. $\boxed{1}$
Correct answer (60). $\boxed{1}$

3. Correct interpretation. $\boxed{1}$
One correct digit. $\boxed{1}$
Correct answer (248). $\boxed{1}$

4. Three correct prime factorisations. $\boxed{1}$
Correct value for one prime. $\boxed{1}$
Correct maximum (42). $\boxed{1}$

5. Correct units digit of the sum (1). $\boxed{1}$
Correct sum of units digits (11). $\boxed{1}$
Substantial progress. $\boxed{1}$
Correct answer (36). $\boxed{1}$

6. A relevant diagram. $\boxed{1}$
Correct location of incentre on an altitude. $\boxed{1}$
Establishing a pair of similar triangles. $\boxed{1}$
Correct answer (671). $\boxed{1}$

7. A correct approach. $\boxed{1}$
Further progress. $\boxed{1}$
Substantial progress. $\boxed{1}$
Correct answer (991). $\boxed{1}$

8. Correct value for GB (12). $\boxed{1}$
Two other useful equations. $\boxed{1}$
A single variable expression for HI. $\boxed{1}$
Correct answer (88). $\boxed{1}$

9. Correct formula for box volume in terms of ball volume. $\boxed{1}$
Correct formula for space volume in terms of ball volume. $\boxed{1}$
Correct formula in two variables for number of balls. $\boxed{1}$
Correct formula in one variable for number of balls. $\boxed{1}$
Correct answer (5). $\boxed{1}$

10. A relevant diagram. $\boxed{1}$
Correct diagonal calculations. $\boxed{1}$
Further progress. $\boxed{1}$
Correct conclusion. $\boxed{1}$

Investigation:

Establishing an upper bound for diameter of the discs. $\boxed{\text{bonus 1}}$
Establishing one disc covers two adjacent square vertices. $\boxed{\text{bonus 1}}$
Correctly establishing the conclusion. $\boxed{\text{bonus 1}}$

The Mathematics Olympiads are supported by the Australian Government
Department of Education, Employment and Workplace Relations
through the Mathematics and Science Participation Program.

ĐỀ THI VÀ ĐÁP ÁN CHI TIẾT

CUỘC THI VÔ ĐỊCH TOÁN CẤP TRUNG HỌC ÚC MỞ RỘNG NĂM 2012

Australian Mathematical Olympiad Committee

A department of the Australian Mathematics Trust

Australian Intermediate Mathematics Olympiad 2012

Questions

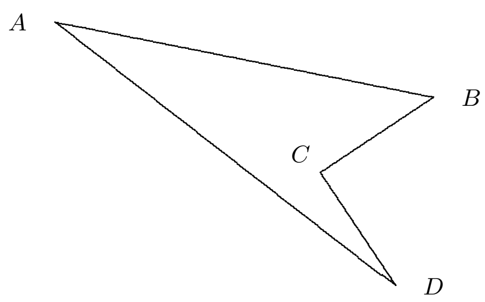
1. Each letter in the grid represents a positive integer.

31	B	C	D	E	7	G	H	I

The sum of any three consecutive integers in the grid is 164. Find the value of H.

[**2 marks**]

2. A yacht race takes place on the course depicted in the diagram below. The starting and finishing point is A, with marker buoys at B, C and D.



The distance AB is 6 km and the distance DA is 6.5 km. Buoys B and C are 2 km apart and buoy C is exactly the halfway mark for the race.

For a yacht moving directly from buoy B to buoy C, its heading would be southwest. For a yacht moving directly from buoy C to buoy D, its heading would be southeast. If the area of water bounded by the course is $R\,\text{km}^2$, find the value of R.

[**2 marks**]

3. Two identical bottles are filled with weak alcohol solutions. In one bottle, the ratio of the volume of alcohol to the volume of water is 1:25. In the other bottle, the ratio of the volume of alcohol to the volume of water is 1:77. If the entire contents of the two bottles are mixed together, the ratio of the volume of alcohol to the volume of water in the mixture is 1:N. Find the value of N.

[**3 marks**]

4. What is the maximum number of terms in a series of consecutive even positive integers whose sum is 1974?

[**3 marks**]

PLEASE TURN OVER THE PAGE FOR QUESTIONS 5, 6, 7, 8, 9 AND 10

5. Let x and y be positive integers satisfying $x^{5x} = y^y$. What is the largest value for x?

[4 marks]

6. The lengths of the sides of triangle T are 190, 323 and 399. What is the length of the shortest altitude of T?

[4 marks]

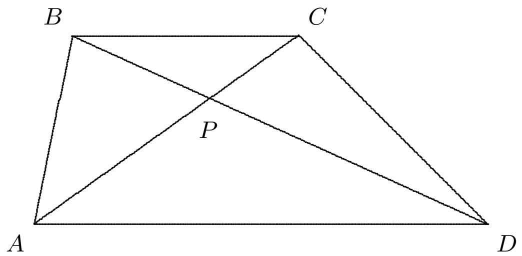
7. The non-zero real numbers x, y, z satisfy the system of equations:

$$\begin{aligned} xy &= 2(x+y) \\ yz &= 3(y+z) \\ zx &= 4(z+x) \end{aligned}$$

Determine $5x + 7y + 9z$.

[4 marks]

8. $ABCD$ is a trapezium with AD parallel to BC. The area of $ABCD$ is 225. The area of $\triangle BPC$ is 49. What is the area of $\triangle APD$?



[4 marks]

9. The nth triangular number is the sum of the first n positive integers. Let T_n denote the sum of the first n triangular numbers. Derive a formula for T_n and, hence or otherwise, prove that $T_n + 4T_{n-1} + T_{n-2} = n^3$.

[5 marks]

10. A bag contains a certain number of 5-cent, 10-cent, 20-cent, 50-cent and one-dollar coins of more than one denomination. For example, if the bag contains nine 5-cent coins and one 20-cent coin, then it contains exactly ten coins and two denominations. Notice in this example that, if any single coin is removed from the bag, then the remaining coins may be divided into three heaps of equal value. Determine all possible combinations of distinct *denominations* the bag may contain so that after removing any coin from the bag, its contents can be divided into three heaps of equal value.

[4 marks]

Investigation

Suppose the bag contains a certain number of 5-cent, 10-cent, 20-cent, 50-cent, one-dollar, and two-dollar coins of more than one denomination. Determine all possible combinations of distinct denominations the bag may contain so that after removing any coin from the bag, its contents can be divided into two heaps of equal value.

[4 bonus marks]

Australian Government

The Australian Mathematics and Science Olympiads are supported by the Australian Government
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Australian Mathematical Olympiad Committee

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

Australian Intermediate Mathematics Olympiad 2012

Solutions

1. *Method 1*

$31 + B + C = 164$ and $B + C + D = 164$, so $D = 31$.
$E + 7 + G = 164$ and $7 + G + H = 164$, so $E = H$. $\boxed{1}$

$D + E + 7 = 164$ and $D = 31$, so $H = E = 164 - 7 - 31 = \mathbf{126}$. $\boxed{1}$

Method 2

The sum of any three consecutive entries in the grid is a constant, so we have the following pattern

x	y	z	x	y	z	x	y	z

$\boxed{1}$

Given that $x = 31$, $z = 7$, and $x + y + z = 164$, we have $H = y = 164 - 31 - 7 = \mathbf{126}$. $\boxed{1}$

2. From the information about distances (C is halfway), $CD = 8 - 6.5 = 1.5$ km.

From the information about headings, $\angle BCD$ is a right angle.
In $\triangle BCD$, $BD = 2.5$ km (3-4-5 triangle, or use Pythagoras' Theorem directly). $\boxed{1}$

Method 1

In $\triangle ABD$, $\angle ABD$ is a right angle (converse of Pythagoras' Theorem in 5-12-13 triangle).
Area of water = area $\triangle ABD$ − area $\triangle BCD = \frac{1}{2}(6 \times 2.5 - 2 \times 1.5) = 6\,\text{km}^2$. So $R = \mathbf{6}$. $\boxed{1}$

Method 2

Half the perimeter of $\triangle ABD$ is 7.5 km. From Heron's formula, the square of the area of $\triangle ABD$ is $7.5(7.5-6)(7.5-6.5)(7.5-2.5) = 7.5(1.5)(1)(5) = 7.5^2$.

Half the perimeter of $\triangle BCD$ is 3 km. From Heron's formula, the square of the area of $\triangle BCD$ is $3(3-2.5)(3-2)(3-1.5) = 3(0.5)(1)(1.5) = 1.5^2$.

Area of water = area $\triangle ABD$ − area $\triangle BCD = 7.5 - 1.5 = 6\,\text{km}^2$. So $R = \mathbf{6}$. $\boxed{1}$

3. Let V be the volume of one bottle.
The volume of alcohol in the first bottle is $\frac{1}{26}V$.
The volume of alcohol in the second bottle is $\frac{1}{78}V$. $\boxed{1}$

The proportion of alcohol by volume in the combined mixture is
$(\frac{1}{26}V + \frac{1}{78}V)/2V = \frac{1}{52}(1 + \frac{1}{3}) = \frac{1}{52} \times \frac{4}{3} = \frac{1}{39}$. $\boxed{1}$

Thus the ratio of volume of alcohol to volume of water is 1:38. Hence $N = \mathbf{38}$. $\boxed{1}$

4. *Method 1*

Let n be the number of terms in the series.

If the series starts with 2, then $1974 = \frac{n}{2}(2 + 2n) = n(n + 1)$.
This has no integer solution since $43 \times 44 = 1892 < 1974$ and $44 \times 45 = 1980 > 1974$. $\boxed{1}$

If the series starts with 4, then $1974 = \frac{n}{2}(4 + 2n + 2) = n(n + 3)$.
This has no integer solution since $42 \times 45 = 1890 < 1974$ and $43 \times 46 = 1978 > 1974$. $\boxed{1}$

If the series starts with 6, then $1974 = \frac{n}{2}(6 + 2n + 4) = n(n + 5)$.
This has integer solution $n = 42$.

If the first term is increased, then the number of terms must decrease to get the same sum.
So the maximum number of terms is **42**. $\boxed{1}$

Method 2

Let n be the number of terms in the series.
We have $1974 = 2a + (2a + 2) + \cdots + (2a + 2n - 2) = \frac{n}{2}(4a + 2n - 2) = n(2a + n - 1)$. $\boxed{1}$

Now $1974 = 2 \times 3 \times 7 \times 47$.
So n is one of the factors 1, 2, 3, 6, 7, 14, 21, 42, 47, ... $\boxed{1}$

As n increases, $2a + n - 1$ decreases and hence a decreases.
If $n = 42$ then $2a + n - 1 = 1974 \div 42 = 47$, hence $2a = 6$.
If $n = 47$ then $2a + n - 1 = 1974 \div 47 = 42$, hence $2a$ is negative.
So the maximum number of terms is **42**. $\boxed{1}$

Method 3

Let n be the number of terms in the series and let A be their average.

Then $nA = 1974 = 2 \times 3 \times 7 \times 47$. $\boxed{1}$

So n is one of the factors 1, 2, 3, 6, 7, 14, 21, 42, 47, ...

Now, if n is odd then A is the middle term in the series and if n is even then A is the odd integer between the two middle terms of the series. $\boxed{1}$

If $n = 42$, then $A = 47$ and the first term is $47 + 1 - 21(2) = 6$.
If $n = 47$, then $A = 42$ and the first term is $42 - 23(2) < 0$.
As n increases, A decreases, hence the first term remains negative.

So the maximum number of terms is **42**. $\boxed{1}$

5. *Method 1*

Clearly $x = 1$ satisfies the given equation.

Suppose $x > 1$ and $x^{5x} = y^y$. (*)

If $y \le x$, then $y^y \le x^x < x^{5x}$, which is a contradiction. Hence $x < y$.

Now suppose $5x \le y$. Then $y^y \ge (5x)^{5x} = 5^{5x}x^{5x}$, again a contradiction. So $x < y < 5x$. $\boxed{1}$

Let p be any prime factor of x. By (*), p divides y.
So $x = p^a x'$ and $y = p^b y'$, where a and b are positive integers and p divides neither x' nor y'.
By (*), $5xa = yb$, so that $a = \frac{y}{5x}b < b$. Hence p^a divides p^b.
Since p is an arbitrary prime factor of x, x divides y. $\boxed{1}$

It follows that $y = cx$, with $c = 2$, 3, or 4.
Feeding this back into (*), we see that $x^{5x} = (cx)^{cx}$, whence $x^5 = (cx)^c$, or $x^{5-c} = c^c$.
If $c = 2$, then $x^3 = 4$, which is impossible.
If $c = 3$, then $x^2 = 27$, again impossible.
If $c = 4$, then $x = 4^4 = 256$. $\boxed{1}$

Now $256^{5\times256} = 2^{8\times5\times256} = 2^{10\times1024} = 1024^{1024}$.
Thus $x = 256$ is the only value for x, other than 1, that satisfies $x^{5x} = y^y$.
Hence the largest value for x is **256**. $\boxed{1}$

Method 2

Clearly $x = 1$ satisfies the given equation. Suppose $x > 1$.

Then $x < y$, otherwise $y^y \le x^x < x^{5x}$, which is a contradiction.

Since $y^y = x^{5x}$, x and y have the same prime factors.

Let $x = p_1^{s_1}p_2^{s_2}\cdots$ and $y = p_1^{t_1}p_2^{t_1}\cdots$. $\boxed{1}$

Then $5s_ix = t_iy$ for all i. So $t_i = ks_i$ for all i where k is a constant.

Since $x < y$, $k > 1$.

Suppose x has at least two distinct primes.

Then $5 > y/x \ge p_1^{t_1-s_1}p_2^{t_1-s_1} \ge (2)(3) = 6$, which is a contradiction.

Therefore $x = p^s$ and $y = p^t$, where p is prime and $s < t$. $\boxed{1}$

We have $5sp^s = tp^t$ and $5s = tp^{t-s} > sp^{t-s}$, hence $5 > p^{t-s}$.

Therefore $p = 3$ and $t - s = 1$ or $p = 2$ and $t - s = 1$ or 2.

If $p = 3$ and $t - s = 1$, then $5s = 3t = 3s + 3$ and s is not an integer.

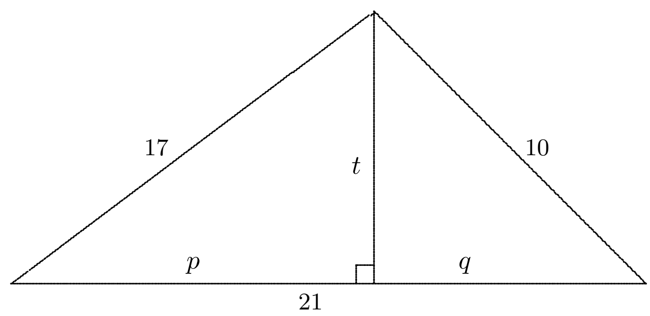
If $p = 2$ and $t - s = 1$, then $5s = 2t = 2s + 2$ and s is not an integer.

If $p = 2$ and $t - s = 2$, then $5s = 4t = 4s + 8$, $s = 8$, $x = 2^8$, and $y = 2^{10}$. $\boxed{1}$

Since $(2^8)^{5\times2^8} = 2^{5\times2^{11}} = (2^{10})^{2^{10}}$, the largest value for x is $2^8 =$ **256**. $\boxed{1}$

6. *Method 1*

Since $190 = 19 \times 10$, $323 = 19 \times 17$, and $399 = 19 \times 21$, we first rescale triangle T by dividing all lengths by 19. This simplifies the calculations. The shortest altitude in a triangle is perpendicular to its longest side. Let t be the length of the shortest altitude and divide the longest side into lengths p and q as shown.



$\boxed{1}$

Alternative 1

Pythagoras gives

$17^2 = p^2 + t^2$ and $10^2 = q^2 + t^2 = (21 - p)^2 + t^2 = 21^2 - 42p + p^2 + t^2$. $\boxed{1}$

Hence $42p = 17^2 + 21^2 - 10^2 = 289 + 441 - 100 = 630$.

So $p = 15$ and $t^2 = 17^2 - 15^2 = 64$. $\boxed{1}$

Thus $t = 8$. Hence the shortest altitude in T is $8 \times 19 = \mathbf{152}$. $\boxed{1}$

Alternative 2

Let x be the angle in the bottom-right corner of T.
From the cosine rule, $17^2 = 10^2 + 21^2 - 2(10)(21)\cos x$. $\boxed{1}$

Hence $\cos x = (100 + 441 - 289)/420 = 252/420 = 21/35 = 3/5$. So $q = 10 \cos x = 6$.

Pythagoras gives $t^2 = 100 - 36 = 64$. $\boxed{1}$

Thus $t = 8$. Hence the shortest altitude in T is $8 \times 19 = \mathbf{152}$. $\boxed{1}$

Method 2

Half the perimeter of T is $(190 + 323 + 399)/2 = 912/2 = 456$. $\boxed{1}$

From Heron's formula, the square of the area of triangle T is
$456(456 - 190)(456 - 323)(456 - 399) = 456(266)(133)(57) = 16(57)(57)(133)(133)$. $\boxed{1}$

The shortest altitude in a triangle is perpendicular to its longest side.
If t is the length of the shortest altitude of T, then the area of T is $\frac{1}{2}(399)t = 4(57)(133)$. $\boxed{1}$

So $3t = 8(57)$ and $t = 8(19) = \mathbf{152}$. $\boxed{1}$

7. *Method 1*

We rewrite the system as:

$1 = 2(\frac{1}{x} + \frac{1}{y}), \quad 1 = 3(\frac{1}{y} + \frac{1}{z}), \quad 1 = 4(\frac{1}{z} + \frac{1}{x}).$ $\boxed{1}$

Put $X = \frac{1}{x}$, $Y = \frac{1}{y}$, $Z = \frac{1}{z}$. Then

$X + Y = \frac{1}{2}, \quad Y + Z = \frac{1}{3}, \quad Z + X = \frac{1}{4}.$ $\boxed{1}$

Adding these equations yields

$X + Y + Z = \frac{1}{2}(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) = \frac{13}{24}.$

Hence

$X = \frac{13}{24} - \frac{1}{3} = \frac{5}{24}, \quad Y = \frac{13}{24} - \frac{1}{4} = \frac{7}{24}, \quad Z = \frac{13}{24} - \frac{1}{2} = \frac{1}{24}.$ $\boxed{1}$

Therefore $5x + 7y + 9z = 24 + 24 + 216 = \mathbf{264}$. $\boxed{1}$

Method 2

We rewrite the system as:

$\frac{1}{2} = \frac{1}{x} + \frac{1}{y}, \quad \frac{1}{3} = \frac{1}{y} + \frac{1}{z}, \quad \frac{1}{4} = \frac{1}{z} + \frac{1}{x}.$ $\boxed{1}$

Adding and subtracting these equations gives:

$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{2}{x}$, $x = 2(\frac{12}{6-4+3}) = \frac{24}{5}$ $\boxed{1}$

$\frac{1}{2} - \frac{1}{4} + \frac{1}{3} = \frac{2}{y}$, $y = 2(\frac{12}{6-3+4}) = \frac{24}{7}$

$\frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{2}{z}$, $z = 2(\frac{12}{4-6+3}) = 24$ $\boxed{1}$

Therefore $5x + 7y + 9z = 24 + 24 + 216 = \mathbf{264}$. $\boxed{1}$

Method 3

We have

$$\begin{aligned} xy &= 2(x+y) \qquad (1) \\ yz &= 3(y+z) \qquad (2) \\ zx &= 4(z+x) \qquad (3) \end{aligned}$$

Since no variable is zero, equation (1) implies $y \neq 2$, equation (2) implies $y \neq 3$, and equation (3) implies $z \neq 4$.

From (1), $x = \dfrac{2y}{y-2}$. From (2), $z = \dfrac{3y}{y-3}$. From (3), $x = \dfrac{4z}{z-4}$. $\boxed{1}$

Hence $\dfrac{2y}{y-2} = \dfrac{4z}{z-4} = \dfrac{\frac{12y}{y-3}}{\frac{3y}{y-3} - 4} = \dfrac{12y}{12-y}$. $\boxed{1}$

So $24y - 2y^2 = 12y^2 - 24y$, $48y = 14y^2$, $y = \frac{24}{7}$.

Then $x = \frac{48}{24-14} = \frac{24}{5}$ and $z = \frac{72}{24-21} = 24$. $\boxed{1}$

Therefore $5x + 7y + 9z = 24 + 24 + 216 = \mathbf{264}$. $\boxed{1}$

Method 4

We have

$$\begin{aligned} xy &= 2(x+y) \quad (1)\\ yz &= 3(y+z) \quad (2)\\ zx &= 4(z+x) \quad (3) \end{aligned}$$

Hence

$$\begin{aligned} xyz &= 2z(x+y) = 2xz+2yz = 8z+8x+6y+6z = 8x+6y+14z \quad (4)\\ xyz &= 3x(y+z) = 3xy+3xz = 6x+6y+12z+12x = 18x+6y+12z \quad (5)\\ xyz &= 4y(z+x) = 4yz+4xy = 12y+12z+8x+8y = 8x+20y+12z \quad (6) \quad \boxed{1} \end{aligned}$$

From (4) and (5), $z = 5x$. From (4) and (6), $z = 7y$. $\boxed{1}$

From (3), $5x^2 = 24x$, $x = 24/5$. From (2), $7y^2 = 24y$, $y = 24/7$, $z = 24$. $\boxed{1}$

Therefore $5x + 7y + 9z = 24 + 24 + 216 = \mathbf{264}$. $\boxed{1}$

8. Let $|\ |$ denote area.

Since AD is parallel to BC, triangles APD and CPB are similar.

Method 1

We have $\dfrac{PD}{PB} = \dfrac{PA}{PC} = \sqrt{\dfrac{|\triangle APD|}{|\triangle CPB|}}$. $\boxed{1}$

$|\triangle PDC| = \dfrac{PD}{PB} \times |\triangle PBC| = \dfrac{\sqrt{|\triangle APD|}}{7} \times 49 = 7 \times \sqrt{|\triangle APD|}$.

$|\triangle PAB| = \dfrac{PA}{PC} \times |\triangle PCB| = \dfrac{\sqrt{|\triangle APD|}}{7} \times 49 = 7 \times \sqrt{|\triangle APD|}$. $\boxed{1}$

Therefore $225 = 49 + 14\sqrt{|\triangle APD|} + |\triangle APD| = (7 + \sqrt{|\triangle APD|})^2$. $\boxed{1}$

So $15 = 7 + \sqrt{|\triangle APD|}$. Hence $|\triangle APD| = (15-7)^2 = \mathbf{64}$. $\boxed{1}$

Method 2

We have $\dfrac{PD}{PB} = \dfrac{PA}{PC} = \dfrac{AD}{CB} = k$, say. $\boxed{1}$

$|\triangle CPD| = \frac{1}{2}CP \times PD \sin\angle CPD = \frac{1}{2}BP \times PA \sin\angle BPA = |\triangle BPA| = R$, say. $\boxed{1}$

Let $|\triangle APD| = T$.

We have: $225 = 49 + T + 2R$,
$T/R = AP/PC = k$,
$T/49 = (\frac{1}{2}AP \times PD \sin\angle APC)/(\frac{1}{2}BP \times PC \sin\angle BPC) = k^2$.

Hence $225 = 49 + 49k^2 + 2(49k) = 49(1 + 2k + k^2) = 49(1+k)^2$. $\boxed{1}$

So $k = 15/7 - 1 = 8/7$. Hence $T = 49(8/7)^2 = \mathbf{64}$. $\boxed{1}$

9. Part 1. First we find a formula for T_n.

Method 1

$$T_n = 1 + 3 + 6 + \cdots + \frac{n}{2}(n+1)$$

$$\begin{aligned}
2T_n &= 2 + 6 + 12 + \cdots + n(n+1) \\
&= (1)(1+1) + (2)(2+1) + (3)(3+1) + \cdots + (n)(n+1) && \boxed{1} \\
&= (1 + 4 + 9 + \cdots + n^2) + (1 + 2 + 3 + \cdots + n) \\
&= \frac{n}{6}(2n+1)(n+1) + \frac{n}{2}(n+1) && \boxed{1} \\
&= \frac{n(n+1)(2n+1+3)}{6}
\end{aligned}$$

So $T_n = \dfrac{n(n+1)(n+2)}{6}$. $\boxed{1}$

Method 2

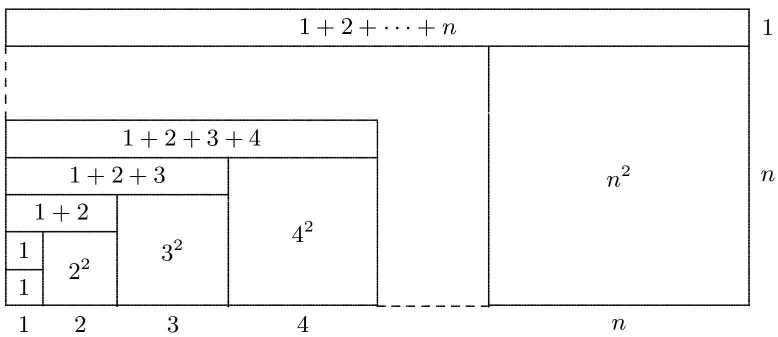
$$T_n = 1 + 3 + 6 + \cdots + \frac{n}{2}(n+1)$$

$$\begin{aligned}
6T_n &= 6 + 18 + 36 + \cdots + 3n(n+1) \\
&= (2^3 - 1^3 - 1) + (3^3 - 2^3 - 1) + (4^3 - 3^3 - 1) + \cdots + ((n+1)^3 - n^3 - 1) && \boxed{1} \\
&= (2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + \cdots + ((n+1)^3 - n^3) - n \\
&= (n+1)^3 - n - 1 && \boxed{1} \\
&= (n+1)((n+1)^2 - 1) = (n+1)n(n+2)
\end{aligned}$$

So $T_n = \dfrac{n(n+1)(n+2)}{6}$. $\boxed{1}$

Method 3

We represent triangular numbers by rectangles of width 1. Packing these rectangles with squares gives another rectangle. The area of this rectangle gives a formula for T_n.



$\boxed{1}$

From the area of the large rectangle, we can see that

$(1^2 + 2^2 + 3^2 + \cdots + n^2) + T_n = (1 + 2 + \cdots + n)(n+1) = \frac{n}{2}(n+1)(n+1)$. $\boxed{1}$

$$\begin{aligned}
\text{Hence } T_n &= \tfrac{n}{2}(n+1)(n+1) - \tfrac{n}{6}(n+1)(2n+1) \\
&= \tfrac{n}{6}(n+1)(3n+3-2n-1) \\
&= \tfrac{n}{6}(n+1)(n+2). && \boxed{1}
\end{aligned}$$

Method 4

The nth triangular number is $\frac{n}{2}(n+1)$, a quadratic polynomial in n. This suggests that T_n is a cubic polynomial in n, that is, $T_n = an^3 + bn^2 + cn + d$ where a, b, c, d are constants that we have to calculate.

We have:

$$\begin{array}{rclcll} T_1 & = & a+b+c+d & = & 1 & (1) \\ T_2 & = & 8a+4b+2c+d & = & 4 & (2) \\ T_3 & = & 27a+9b+3c+d & = & 10 & (3) \\ T_4 & = & 64a+16b+4c+d & = & 20 & (4) \end{array} \qquad \boxed{1}$$

Subtracting equation (3) from (4), then (2) from (3), then (1) from (2) gives:

$$\begin{array}{rcll} a+b+c+d & = & 1 & (5) \\ 7a+3b+c+0 & = & 3 & (6) \\ 26a+8b+2c+0 & = & 9 & (7) \\ 63a+15b+3c+0 & = & 19 & (8) \end{array}$$

Subtracting equation (6) from (7) twice, then (6) from (8) three times gives:

$$\begin{array}{rcll} a+b+c+d & = & 1 & (9) \\ 7a+3b+c+0 & = & 3 & (10) \\ 12a+2b+0+0 & = & 3 & (11) \\ 42a+6b+0+0 & = & 10 & (12) \end{array}$$

Subtracting equation (11) from (12) three times gives:

$$\begin{array}{rcll} a+b+c+d & = & 1 & (13) \\ 7a+3b+c+0 & = & 3 & (14) \\ 12a+2b+0+0 & = & 3 & (15) \\ 6a+0+0+0 & = & 1 & (16) \end{array} \qquad \boxed{1}$$

From (16) $a = \frac{1}{6}$. From (15) $b = \frac{3-2}{2} = \frac{1}{2}$.
From (14) $c = 3 - \frac{7}{6} - \frac{3}{2} = \frac{1}{3}$. From (13) $d = 1 - \frac{1}{3} - \frac{1}{2} - \frac{1}{6} = 0$.
So $T_n = \dfrac{n^3}{6} + \dfrac{n^2}{2} + \dfrac{n}{3} = \dfrac{n(n^2+3n+2)}{6} = \dfrac{n(n+1)(n+2)}{6}$.
We know from our calculations that this formula gives the first four tetrahedral numbers. To prove it gives all tetrahedral numbers, we show that the difference $T_n - T_{n-1}$ is the nth triangular number:

$$T_n - T_{n-1} = \frac{n(n+1)(n+2)}{6} - \frac{(n-1)(n)(n+1)}{6} = \frac{(n)(n+1)(n+2-n+1)}{6} = \frac{n(n+1)}{2}.$$

$\boxed{1}$

Method 5

The nth triangular number is $\frac{n}{2}(n+1)$, a quadratic polynomial in n. This suggests that T_n is a cubic polynomial in n, that is, $T_n = an^3 + bn^2 + cn + d$ where a, b, c, d are constants that we have to calculate.

We have:

$T_n - T_{n-1} = \frac{n}{2}(n+1)$.

Therefore:

$$\begin{aligned}
\tfrac{1}{2}n^2 + \tfrac{1}{2}n &= an^3 + bn^2 + cn + d - (a(n-1)^3 + b(n-1)^2 + c(n-1) + d) \\
&= an^3 + bn^2 + cn - (a(n^3 - 3n^2 + 3n - 1) + b(n^2 - 2n + 1) + c(n-1)) \\
&= 3an^2 - 3an + a + 2bn - b + c
\end{aligned}$$ $\boxed{1}$

Equating corresponding coefficients of powers of n we get:

$3a = \frac{1}{2}$, $-3a + 2b = \frac{1}{2}$, $a - b + c = 0$.

So $a = \dfrac{1}{6}$, $b = \frac{1}{2}(\frac{1}{2} + \frac{1}{2}) = \frac{1}{2}$, $c = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$. $\boxed{1}$

Hence $T_n = \dfrac{n^3}{6} + \dfrac{n^2}{2} + \dfrac{n}{3} + d$.

Therefore $T_1 = \dfrac{1}{6} + \dfrac{1}{2} + \dfrac{1}{3} + d = 1 + d$. Since $T_1 = 1$, $d = 0$.

So $T_n = \dfrac{n^3}{6} + \dfrac{n^2}{2} + \dfrac{n}{3} = \dfrac{n}{6}(n^2 + 3n + 2) = \dfrac{n(n+1)(n+2)}{6}$. $\boxed{1}$

Method 6

From the construction of Pascal's Triangle, the second column gives n, the third column gives the nth triangular number, and the fourth column gives the sum of the first n triangular numbers. $\boxed{1}$

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4	1					
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	46	70	46	28	8	1	
1	9	36	74	116	116	74	36	9	1

$\boxed{1}$

Thus $T_n = \binom{n+2}{3} = \dfrac{n(n+1)(n+2)}{6}$. $\boxed{1}$

Part 2. Next we prove that $T_n + 4T_{n-1} + T_{n-2} = n^3$.

Method 1

$$\begin{aligned} T_n + 4T_{n-1} + T_{n-2} &= \frac{n(n+1)(n+2)}{6} + 4\frac{(n-1)(n)(n+1)}{6} + \frac{(n-2)(n-1)(n)}{6} && \boxed{1} \\ &= \frac{n}{6}((n+1)(n+2) + 4(n-1)(n+1) + (n-2)(n-1)) \\ &= \frac{n}{6}(6n^2 + 3n + 2 - 4 - 3n + 2) = n^3 && \boxed{1} \end{aligned}$$

Method 2

This is a geometric proof. We represent the $5 \times 5 \times 5$ cube by five 5×5 horizontal layers, top layer first and bottom layer last. The numbers represent the six required tertrahedra: tetrahedon 1 is T_3, tetrahedra 2 to 5 are T_4, and tetrahedron 6 is T_5.
Thus $T_5 + 4T_4 + T_3 = 5^3$. $\boxed{1}$

6	6	6	6	6
5	5	5	5	4
5	5	5	4	4
5	5	4	4	4
5	4	4	4	4

6	6	6	6	3
6	6	6	6	2
5	5	5	4	1
5	5	4	4	1
5	4	4	4	1

6	6	6	3	3
6	6	6	3	2
6	6	6	2	2
5	5	4	1	1
5	4	4	1	1

6	6	3	3	3
6	6	3	3	2
6	6	3	2	2
6	6	2	2	2
5	4	1	1	1

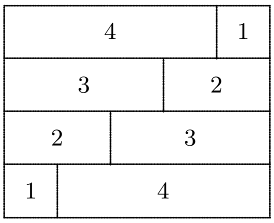
6	3	3	3	3
6	3	3	3	2
6	3	3	2	2
6	3	2	2	2
6	2	2	2	2

The diagrams clearly generalise for all $n \geq 3$. $\boxed{1}$

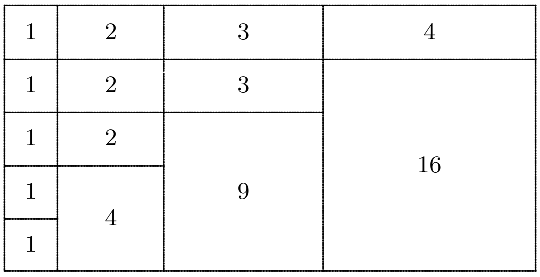
Comment

It is assumed in some of these solutions that students know the formula for the nth triangular number (the sum of the first n positive integers) or the sum of the first n squares. If necessary, these can be established in various ways. For example, here is a geometrical proof for each.

First consider summing the first n integers. Each number k is represented by a $k \times 1$ rectangle. Take $n = 4$ as an example. From the area of the large rectangle in the following diagram, we can see that $2(1 + 2 + 3 + 4) = 4(4 + 1)$. For general n we get $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$.



In a similar way we can find a formula for the sum of the first n squares. Again, take $n = 4$ as an example.



From the area of the large rectangle, we can see that

$(1 + 2 + 3 + 4)(4 + 1) = (1^2 + 2^2 + 3^2 + 4^2) + (1) + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4)$.

In general,

$(1 + 2 + \cdots + n)(n + 1) = (1^2 + 2^2 + \cdots + n^2) + (1) + (1 + 2) + \cdots + (1 + 2 + \cdots + n)$.

Thus $\frac{1}{2}n(n + 1)^2 = (1^2 + 2^2 + \cdots + n^2) + 1 + \frac{1}{2}2(2 + 1) + \cdots + \frac{1}{2}n(n + 1)$.

Hence
$$\begin{aligned} n(n + 1)^2 &= 2(1^2 + 2^2 + \cdots + n^2) + (1 + 1) + (2^2 + 2) + \cdots + (n^2 + n) \\ &= 3(1^2 + 2^2 + \cdots + n^2) + (1 + 2 + \cdots + n) \\ &= 3(1^2 + 2^2 + \cdots + n^2) + \tfrac{1}{2}n(n + 1). \end{aligned}$$

So
$$\begin{aligned} 6(1^2 + 2^2 + \cdots + n^2) &= 2n(n + 1)^2 - n(n + 1) \\ &= n(n + 1)(2(n + 1) - 1) \\ &= n(n + 1)(2n + 1). \end{aligned}$$

Finally $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$.

10. Let the total value of the contents of the bag be S cents. Remove one coin of denomination m cents, say. Then the total value of the coins left in the bag is $S - m$ cents. This amount must be divisible by 3. Similarly, if one coin of denomination n cents is removed, $S - n$ must be divisible by 3. Hence $(S - n) - (S - m) = m - n$ must also be divisible by 3. So any two denominations occurring in the bag differ by a multiple of 3, hence each must have the same remainder when divided by 3. $\boxed{1}$

The available denominations are: 5, 10, 20, 50, 100 cents. They fall into two classes according to their remainders after division by 3: $\{5, 20, 50\}$ and $\{10, 100\}$. $\boxed{1}$

The following table shows, with examples, all combinations of denominations that could have occurred in the bag.

Denominations	Possible coins	One coin removed	Three equal heaps
5, 20	1×20, 9×5	9×5	$3 \times (3 \times 5)$
		1×20, 8×5	1×20, 4×5, 4×5
5, 50	1×50, 21×5	21×5	$3 \times (7 \times 5)$
		1×50, 20×5	1×50, 10×5, 10×5
20, 50	4×50, 6×20	3×50, 6×20	$3 \times (1 \times 50 + 2 \times 20)$
		4×50, 5×20	2×50, 2×50, 5×20
5, 20, 50	2×50, 2×20, 30×5	1×50, 2×20, 30×5	$1 \times 50 + 1 \times 20 + 2 \times 5$, $1 \times 20 + 12 \times 5$, 16×5
		2×50, 1×20, 30×5	$1 \times 50 + 1 \times 20 + 4 \times 5$, $1 \times 50 + 8 \times 5$, 18×5
		2×50, 2×20, 29×5	$1 \times 50 + 1 \times 20 + 5 \times 5$, $1 \times 50 + 1 \times 20 + 5 \times 5$, 19×5
10, 100	1×100, 21×10	21×10	$3 \times (7 \times 10)$
		1×100, 20×10	1×100, 10×10, 10×10

$\boxed{2}$

Investigation

Let the total value of the contents of the bag be S units. Remove one coin of denomination m units, say. Then the total value of the coins left in the bag is $S - m$ units. This amount must be divisible by 2. Similarly, if one coin of denomination n units is removed, $S - n$ must be divisible by 2. Hence $(S - n) - (S - m) = m - n$ must also be divisible by 2. So all the denominations in the bag must be even or all must be odd. $\boxed{\text{bonus 1}}$

Since there are at least two denominations in the bag and the only odd denomination is 5 cents, there cannot be any 5 cent coins in the bag. This leaves us with the following set of denominations: 10, 20, 50, 100 and 200 cents. All of them are multiples of 10, so 10 cents is our new unit. Call it a *bob*, both singular and plural. The denominations are now 1, 2, 5, 10 and 20 bob. Hence the denominations in the bag will be 1 bob and 5 bob (both odd), or at least two of 2, 10, and 20 bob (all even). $\boxed{\text{bonus 1}}$

In the latter case all three denominations are multiples of two, so our new unit is *two-bob* (originally 20 cents). Thus each coin in this bag will be worth 1, 5, or 10 two-bobs. Since 10 is the only even number amongst 1, 5, and 10, it cannot be a denomination occurring in the bag.

Hence the denominations in the bag will be 1 bob and 5 bob (i.e. 10 cents and 50 cents) or 1 two-bob and 5 two-bobs (i.e. 20 cents and 100 cents). Here are some examples to show both are possible. $\boxed{\text{bonus 1}}$

Denominations	Possible coins	One coin removed	Two equal heaps
10, 50	1×50, 6×10	6×10	$2 \times (3 \times 10)$
		1×50, 5×10	1×50, 5×10
20, 100	1×100, 6×20	6×20	$2 \times (3 \times 20)$
		1×100, 5×20	1×100, 5×20

$\boxed{\text{bonus 1}}$

Marking Scheme

1. A correct approach. $\boxed{1}$
Correct answer (126). $\boxed{1}$

2. Correct distance for BD (2.5 km). $\boxed{1}$
Correct answer (6). $\boxed{1}$

3. Correct proportions of alcohol in two vessels ($V/26$, $V/78$). $\boxed{1}$
Correct proportion of alcohol in combined mixture (1/39). $\boxed{1}$
Correct value for N (38). $\boxed{1}$

4. A correct approach. $\boxed{1}$
A correct list of options or cases for n. $\boxed{1}$
Correct maximum (42). $\boxed{1}$

5. A correct approach. $\boxed{1}$
Substantial progress. $\boxed{1}$
Completion of cases. $\boxed{1}$
Correct maximum value (256). $\boxed{1}$

6. A relevant diagram or correct semiperimeter. $\boxed{1}$
A useful formula correctly applied. $\boxed{1}$
Another useful formula correctly applied. $\boxed{1}$
Correct answer (152). $\boxed{1}$

7. A relevant system of equations. $\boxed{1}$
A useful set of substitutions. $\boxed{1}$
Correct values for x, y, z (24/5, 24/7, 24). $\boxed{1}$
Correct answer (264). $\boxed{1}$

8. A correct approach. $\boxed{1}$
Establishing areas of triangles CPD and BPA equal. $\boxed{1}$
Relevant equation in one variable. $\boxed{1}$
Correct area of triangle APD (64). $\boxed{1}$

9. A correct approach. $\boxed{1}$
Substantial progress. $\boxed{1}$
Correct formula for T_n $(n(n+1)(n+2)/6)$. $\boxed{1}$
A correct approach to proving $T_n + 4T_{n-1} + T_{n-2} = n^3$. $\boxed{1}$
A convincing proof. $\boxed{1}$

10. Establishing that any two denominations are congruent modulo 3. $\boxed{1}$
Two correct congruency classes. $\boxed{1}$
Demonstrating three denomination combinations are possible. $\boxed{1}$
Demonstrating the remaining two denomination combinations are possible. $\boxed{1}$

Investigation:

Establishing that any two denominations are congruent modulo 2. $\boxed{\text{bonus } 1}$
Two correct congruency classes after dividing by 10. $\boxed{\text{bonus } 1}$
Two correct congruency classes after further dividing by 2. $\boxed{\text{bonus } 1}$
Demonstrating the two denomination combinations are possible. $\boxed{\text{bonus } 1}$

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